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## UNIQUENESS FOR THE SKOROKHOD EQUATION WITH NORMAL REFLECTION IN LIPSCHITZ DOMAINS

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**AMS subject classification:** Primary 60J60, Secondary 60J50

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# Uniqueness for the Skorokhod equation with normal reflection in Lipschitz domains

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## 1. Introduction.

We consider the Skorokhod equation in a domain  $D \subseteq \mathbb{R}^d$ ,  $d \geq 2$ :

$$(1.1) \quad dX_t = dW_t + \frac{1}{2}\nu(X_t) dL_t, \quad X_0 = x_0,$$

where  $W_t$  is  $d$ -dimensional Brownian motion,  $L_t$  is the local time of  $X_t$  on the boundary of  $D$ , and  $\nu$  is the inward pointing unit normal vector. It is well-known that in smooth domains  $X_t$  is reflecting Brownian motion with normal reflection.

There are various types of solutions to (1.1). Pathwise existence and uniqueness holds for (1.1) when the domain  $D$  is a  $C^2$  domain. This was proved by Lions and Sznitman [LS]. In fact they considered domains slightly more general than  $C^2$ , but the class of domains they considered does not contain the class of  $C^{1+\alpha}$  domains for any  $\alpha \in (0, 1)$ . They also considered more general diffusion coefficients and considered oblique reflection as well as normal reflection. Their work was generalized by Dupuis and Ishii [DI], who considered  $C^1$  domains, but required the angle of reflection to vary in almost a  $C^2$  manner. For normal reflection, this implies the domains must be nearly  $C^2$ .

Another type of uniqueness is weak uniqueness. That means that there exist processes  $X_t$  and  $W_t$  satisfying (1.1) where  $W_t$  is a Brownian motion, but that  $X$  need not be measurable with respect to the  $\sigma$ -fields generated by  $W$ . In [BH1] reflecting Brownian motion in bounded Lipschitz domains with normal reflection was constructed using Dirichlet forms, and in [BH2] and [FOT], Ex. 5.2.2, it was shown that this process provides a weak solution to the Skorokhod equation. These results were extended in [FT].

Closely related to weak uniqueness is the submartingale problem of Stroock and Varadhan [SV1]. They proved existence and uniqueness of the submartingale problem corresponding to (1.1) with more general diffusion coefficients and with oblique reflection for  $C^2$  domains.

Using Dirichlet forms techniques, Williams and Zheng [WZ] constructed reflecting Brownian motion that provides a weak solution to (1.1) for domains more irregular than Lipschitz domains. Further research along these lines was done by [CFW] and [C]. Uniqueness of reflecting Brownian motion corresponding to the Dirichlet form for Brownian motion can be proved for quite general domains by the techniques of [F].

In this paper we prove weak uniqueness of (1.1) for Lipschitz domains. We prove that there is only one probability measure  $\mathbb{P}$  under which  $W_t$  is a  $d$ -dimensional Brownian motion,  $X_t$  spends 0 time on the boundary,  $L_t$  is the local time of  $X_t$  on the boundary (defined as a limit of occupation times), and (1.1) holds. See Theorem 2.2 for a precise statement.

The question of weak uniqueness is a natural one. In problems of weak convergence, (e.g., in proving convergence of penalty methods as in [LS]) one is led to solutions to the Skorokhod equation. If one knew *a priori* that the solution was associated to a Dirichlet

form, the uniqueness would be easy, but in general one does not know in advance that the solution corresponds to a Dirichlet form or even that the solution is strong Markov. Submartingale problems are also a natural class to consider, but in Lipschitz domains there is considerable difficulty in formulating them; typically, the class of test functions one would want to consider is empty.

In Section 2 we give definitions and recall a few facts about the reflecting Brownian motion constructed in [BH1]. We also prove a few preliminary propositions.

The reason problems of weak uniqueness tend to be hard is the paucity of the right type of functions; this is also the reason problems involving Lipschitz domains are typically much harder than those involving smoother domains. Section 3 is devoted to constructing a sequence of functions satisfying certain conditions. An estimate of Dahlberg on harmonic measure and one of Jerison and Kenig for solutions to the Neumann problem play key roles.

Section 4 contains the proof of weak uniqueness for (1.1) for Lipschitz domains. The main idea is to show that any two solutions must have the same potentials.

In Section 5 we pose a question about the existence of strong solutions. An affirmative answer would imply that in fact pathwise uniqueness holds for (1.1) in Lipschitz domains. At the present time pathwise uniqueness is not known even for  $C^{1+\alpha}$  domains in the plane.

## 2. Preliminaries.

**Notation.** We let  $B(x, r)$  denote the open ball of radius  $r$  centered at  $x$ . The letter  $c$  with subscripts will denote constants; we begin renumbering anew at each proposition or theorem. Points  $x = (x_1, \dots, x_d)$  will sometimes be written  $(\tilde{x}, y)$ , where  $\tilde{x} = (x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}$  and  $y = x_d$ . We will also use polar coordinates:  $x = (r, \theta)$ , where  $r = |x|$  and  $\theta = x/|x| \in \partial B(0, 1)$ . The inner product in  $\mathbb{R}^d$  of  $x$  and  $y$  is written  $x \cdot y$ .

For a domain  $D$  with  $x \in \partial D$ , the boundary of  $D$ , we let  $\nu(x)$  be the inward pointing normal vector and  $\nu_o(x) = -\nu(x)$  the outward pointing normal vector. We write  $\sigma(dx)$  for surface measure on  $\partial D$ .

**Lipschitz domains.** A function  $f : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  or  $f : \partial B(0, 1) \rightarrow \mathbb{R}$  is Lipschitz if there exists  $M$  such that  $|f(x) - f(y)| \leq M|x - y|$  for all  $x, y$  in the domain of  $f$ . The smallest such  $M$  is the Lipschitz constant of  $f$ . A domain  $D$  is a Lipschitz domain if for all  $z \in \partial D$  there exists a coordinate system  $CS_z$ , an  $r_z > 0$ , and a Lipschitz function  $\Gamma_z$  such that

$$D \cap B(z, r_z) = \{x = (\tilde{x}, y) \text{ in } CS_z : y > \Gamma_z(\tilde{x})\} \cap B(z, r_z),$$

i.e., locally  $D$  looks like the region above the graph of a Lipschitz function. A Lipschitz domain is star-like (relative to 0) if there exists a Lipschitz function  $\varphi : \partial B(0, 1) \rightarrow (0, \infty)$  such that  $D = \{(r, \theta) : 0 \leq r < \varphi(\theta)\}$ .

For each  $z \in \partial D$ , where  $D$  is a star-like Lipschitz domain, we let  $V_\beta(z)$  denote the interior of the convex hull of  $\{z\} \cup B(0, \beta)$ . We fix  $\beta$  small enough so that  $V_{2\beta}(z) \subseteq D$  for all  $z \in \partial D$ . When we need to emphasize the domain we write  $V_\beta^D(z)$ . Let

$$(2.1) \quad U_r = \{x \in D : \text{dist}(x, \partial D) < r\}.$$

If  $u$  is a function on  $D$ , we let

$$N(u)(z) = N^D(u)(z) = \sup_{x \in V_\beta(z)} |u(x)|$$

and

$$N_r(u)(z) = N_r^D(u)(z) = \sup\{|u(x)| : x \in V_\beta(z) \cap U_r\}.$$

$L^2$  norms with respect to surface measure on  $\partial D$  will be denoted  $\|f\|_{2, \partial D}$ . Thus

$$\|f\|_{2, \partial D}^2 = \int_{\partial D} |f(z)|^2 \sigma(dz).$$

We will first prove our results for star-like Lipschitz domains, and then extend them to general Lipschitz domains. Let us describe the special set-up that we first consider.

(2.2) *Let  $D$  be a star-like Lipschitz domain, let  $\rho < (\inf \varphi)/4$ , and let  $K = \overline{B(0, \rho)}$ . We will consider open subsets  $G$  of  $\partial B(0, 1)$  and we consider the corresponding open subsets  $A = \varphi(G)$  of  $\partial D$ :*

$$A = \varphi(G) = \{(r, \theta) : r = \varphi(\theta), \theta \in G\}.$$

**Reflecting Brownian motion.** In this subsection let us suppose the dimension  $d$  is greater than or equal to 3. Let  $D$  be a Lipschitz domain with  $K$  a compact set contained in  $D$  such that  $K$  has smooth boundary. In [BH1] a strong Markov process  $(Q^x, X_t)$ ,  $x \in \overline{D}$ , was constructed that represents reflecting Brownian motion in  $\overline{D}$  with absorption at  $K$ . We recall a few properties and derive some others. See [BH1] for details. Let

$$T_A = T(A) = \inf\{t > 0 : X_t \in A\}.$$

Reflecting Brownian motion in  $D$  has a Green function  $g(x, y)$  that is symmetric in  $x$  and  $y$  for  $x, y \in D - K$ , harmonic in  $y$  in  $D - K - \{x\}$ , harmonic in  $x$  in  $D - K - \{y\}$ , vanishes as  $x$  or  $y$  tends to the boundary of  $K$ , and there exists  $c_1$  depending only on  $D$  and  $K$  such that

$$(2.3) \quad g(x, y) \leq c_1 |x - y|^{2-d}.$$

If  $D$  is star-like, the constant  $c_1$  depends only on  $\rho$ ,  $\|\nabla\varphi\|_\infty$ ,  $\inf\varphi$ , and  $\sup\varphi$ . In particular, for each  $\rho' > 0$ ,  $g(x, \cdot)$  is bounded in  $\overline{D} - K - B(x, \rho')$ .

A consequence of (2.3) is that

$$(2.4) \quad \mathbb{E}^x T_K = \int_{\overline{D}-K} g(x, y) dy \leq c_2, \quad x \in \overline{D}.$$

Another consequence of (2.3) is that

$$\mathbb{E}^x \int_0^{T_K} 1_{\overline{U}_r}(X_s) ds = \int_{\overline{U}_r} g(x, y) dy \rightarrow 0$$

as  $r \rightarrow 0$ , so  $X_t$  spends zero time in  $\partial D$ , and hence starting at  $x \in \partial D$ , the process leaves  $\partial D$  immediately.

In [BH1] it is proved that there exists a continuous additive functional  $L_t$  corresponding to the measure  $\sigma(dy)$ :

$$\mathbb{E}^x L_{T_K} = \int_{\partial D} g(x, y) \sigma(dy), \quad x \in \overline{D},$$

and  $L_t$  increases only when  $X_t$  is in the support of  $\sigma$ , namely  $\partial D$ . It follows from (2.3) that  $\mathbb{E}^x L_{T_K} \leq c_3$ ,  $x \in \overline{D}$ , where  $c_3$  depends on the domain  $D$ . When  $D$  is star-like,  $c_3$  depends on  $\rho$ ,  $\|\nabla\varphi\|_\infty$ ,  $\inf\varphi$ , and  $\sup\varphi$ . Suppose  $f_m$  are nonnegative bounded functions supported in  $D - K$  such that  $f_m(y) dy$  converges weakly to  $\sigma(dy)$  (this is the usual weak convergence of measures in probability theory, except that we do not assume the total mass is one) and also that there exist  $c_4 > 0$  and  $\gamma \in [0, 1)$  such that

$$(2.5) \quad \int_{B(x, s) \cap \overline{D}} f_m(y) dy \leq c_4 (s \wedge 1)^{d-1-\gamma}, \quad x \in D, s > 0.$$

An example of  $f_m$  satisfying (2.5) is  $f_m(y) = a_m^{-1} 1_{U_{1/m}}(y)$ , where  $a_m$  is the Lebesgue measure of  $U_{1/m}$ . If the  $f_m$  satisfy (2.5), we have by the proof of [BK], Section 2, that  $\int g(x, y) f_m(y) dy \rightarrow \int g(x, y) \sigma(dy)$  uniformly in  $x$ . Let

$$(2.6) \quad A_m(t) = \int_0^t f_m(X_s) ds.$$

By [BK], Section 2,

$$\sup_{t \leq T_K} |A_m(t) - L_t| \rightarrow 0$$

in probability as  $m \rightarrow \infty$ .

Suppose  $D$  is star-like,  $\varphi$  is smooth,  $x_0 \in \overline{D}$ , and  $B = \overline{(D - K)} \cap B(x_0, r)$  for some  $r > 0$ . Then  $h(x) = \mathbb{E}^x f(X_{T(B^c)})$  is harmonic in  $D \cap B$  and has normal derivative 0 on

$(\partial D) \cap B$ . There exist  $c_5$  and  $\alpha$  depending only on  $\rho$ , the Lipschitz constant of  $\varphi$ , the supremum and infimum of  $\varphi$ , and  $r$  such that

$$(2.7) \quad |h(x) - h(y)| \leq c_5 |x - y|^\alpha \|f\|_\infty, \quad x, y \in \overline{(D - K)} \cap B(x_0, r/2).$$

Reflecting Brownian motion satisfies a tightness estimate similar to that of ordinary Brownian motion. By [BH1] there exist  $c_6$  and  $c_7$  such that if  $x \in \overline{D}$  and  $r > 0$ ,

$$(2.8) \quad \mathbb{P}^x(\sup_{s \leq t} |X_s - x| \geq \lambda) \leq c_6 e^{-c_7 \lambda^2 / t}.$$

**Lemma 2.1.** *Suppose  $D$  is star-like and  $\varepsilon, \eta > 0$ . There exists  $\delta$  depending only on  $\varepsilon, \eta, \|\nabla \varphi\|_\infty, \sup \varphi$ , and  $\inf \varphi$  such that if  $\text{dist}(x, \partial D) < \delta$ , then*

$$\mathbb{Q}^x(T_{\partial B(x, \eta)} < T_{\partial D}) < \varepsilon.$$

**Proof.** Let  $r = \text{dist}(x, \partial D)$ . Since  $D$  is Lipschitz, there exists  $c_1 > 0$  depending only on the Lipschitz constant of  $\varphi$  such that if  $s \geq r$ , then the surface measure of  $\partial B(x, 2s) \cap D^c$  is greater than  $c_1$  times the surface measure of  $\partial B(x, 2s)$ . Since the law of  $X_t$  up until time  $T_{\partial D}$  is the same as that of standard  $d$ -dimensional Brownian motion and the distribution of Brownian motion on exiting a ball is uniform on the surface of the ball,

$$\mathbb{Q}^x(T_{\partial B(x, 2r)} < T_{\partial D}) \leq 1 - c_1.$$

Any point  $y$  in  $\partial B(x, 2r)$  is a distance  $2r$  from  $x$  and hence no more than  $3r$  from  $D^c$ . So if  $y \in D \cap \partial B(x, 2r)$ , the same reasoning tells us

$$\mathbb{Q}^y(T_{\partial B(y, 6r)} < T_{\partial D}) \leq 1 - c_1.$$

By the strong Markov property and the fact that  $B(y, 6r) \subseteq B(x, 8r)$  if  $|y - x| = 2r$ ,

$$\mathbb{Q}^x(T_{\partial B(x, 8r)} < T_{\partial D}) \leq (1 - c_1)^2.$$

We repeat the argument. A point in  $\partial B(x, 8r)$  is a distance no more than  $9r$  from  $D^c$ , and a ball of radius  $18r$  about such a point is contained in  $B(x, 26r)$ , so using the strong Markov property,

$$\mathbb{Q}^x(T_{\partial B(x, 26r)} < T_{\partial D}) \leq (1 - c_1)^3.$$

We continue by induction and obtain

$$\mathbb{Q}^x(T_{\partial B(x, (3^m - 1)r)} < T_{\partial D}) \leq (1 - c_1)^m.$$

Now choose  $m$  so that  $(1 - c_1)^m < \varepsilon$  and then choose  $\delta$  so that  $(3^m - 1)\delta < \eta$ .  $\square$

**Skorokhod equation.** We now suppose that  $d \geq 2$ . In [BH2] and [FOT], Ex. 5.2.2, it was shown that the  $(\mathbb{Q}^x, X_t)$  constructed in [BH1] satisfy the Skorokhod equation: there exists a  $d$ -dimensional Brownian motion  $W_t$  such that

$$(2.9) \quad dX_t = dW_t + \frac{1}{2}\nu(X_t) dL_t.$$

We want to show that the solution to (2.9) is unique in law. To be precise, let  $D$  be an arbitrary Lipschitz domain. We say that

(2.10) *a probability measure  $\mathbb{P}$  is a solution to the Skorokhod equation (2.9) starting from  $x_0 \in \overline{D}$  if*

$$(a) \quad \mathbb{P}(X_0 = x_0) = 1,$$

$$(b) \quad \int_0^\infty 1_{\partial D}(X_s) ds = 0,$$

(c) *there exist nonnegative functions  $f_m$  with support in  $D$  such that  $f_m(y) dy$  converges weakly to  $\sigma(dy)$ , the  $f_m$  satisfy (2.5), and for each  $t_0$  we have*

$$\sup_{t \leq t_0} |A_m(t) - L_t| \rightarrow 0$$

*in  $\mathbb{P}$ -probability as  $m \rightarrow \infty$ , where the  $A_m$  are defined by (2.6), and*

(d) *there exists a continuous process  $W_t$  which under  $\mathbb{P}$  is a  $d$ -dimensional Brownian motion with respect to the filtration of  $X$  such that for all  $t$ ,  $X_t \in \overline{D}$  and*

$$X_t - X_0 = W_t + \frac{1}{2} \int_0^t \nu(X_s) dL_s.$$

By our discussion above there exists at least one solution to (2.10), namely  $Q^{x_0}$ . Saying that  $W_t$  is a Brownian motion with respect to the filtration generated by  $X_t$  means that  $W_t - W_s$  has the same distribution as that of a normal random variable with mean 0 and variance  $t - s$  and  $W_t - W_s$  is independent of  $\sigma(X_r; r \leq s)$  whenever  $s < t$ .

Our main result is the following.



**Theorem 2.2.** *If  $D$  is a Lipschitz domain in  $\mathbb{R}^d$ ,  $d \geq 2$ , then there is exactly one solution to (2.10).*

The proof of Theorem 2.2 will take up Sections 3 and 4.

The condition (2.10)(c) is slightly stronger than the one sometimes seen in the literature, namely, that  $L_t$  be a nondecreasing continuous process that increases only when  $X_t \in \partial D$ . Here we are essentially requiring the local time  $L_t$  to be an additive functional corresponding to surface measure on the boundary.

We will need the following proposition. Let  $\theta_t$  be shift operators so that  $X_s \circ \theta_t = X_{s+t}$ . By [B], Section I.2, we may always suppose such  $\theta_t$  exist.

**Proposition 2.3.** *Let  $\mathbb{P}$  be a solution to (2.10) started at  $x_0 \in \overline{D}$ , let  $S$  be a finite stopping time, and let  $\mathbb{P}_S(\omega, d\omega')$  be a regular conditional probability for the law of  $X \circ \theta_S$  under  $\mathbb{P}[\cdot | \mathcal{F}_S]$ . Then  $\mathbb{P}$ -almost surely,  $\mathbb{P}_S$  is a solution to (2.10) started at  $X_S(\omega)$ .*

**Proof.** The proof is standard. Let  $A(\omega) = \{\omega' : X_0(\omega') = X_S(\omega)\}$ . Then

$$A(\omega) \circ \theta_S = \{\omega' : X_0 \circ \theta_S(\omega') = X_S(\omega)\} = \{\omega' : X_S(\omega') = X_S(\omega)\}.$$

So

$$\mathbb{P}(A(\omega) \circ \theta_S | \mathcal{F}_S) = 1_{\{X_S(\omega)\}}(X_S) = 1, \quad \text{a.s.}$$

If  $B = \{L_t \text{ is the uniform limit of the } A_m(t)\}$ , then

$$B \circ \theta_S = \{L_{t+S} - L_S \text{ is the uniform limit of } A_m(t+S) - A_m(S)\},$$

and so  $\mathbb{P}(B \circ \theta_S | \mathcal{F}_S) = 1$ , a.s. The proof that the process spends 0 time on the boundary under  $\mathbb{P}_S$  is similar.

Finally, the law of  $[X_t - X_0 - \frac{1}{2} \int_0^t \nu(X_s) dL_s] \circ \theta_S$  given  $\mathcal{F}_S$  is the law of  $[X_{t+S} - X_S - \frac{1}{2} \int_S^{S+t} \nu(X_s) dL_s]$  given  $\mathcal{F}_S$ . This is a Brownian motion by the strong Markov property of Brownian motion.  $\square$

### 3. Some analytic estimates.

We suppose throughout this section that the dimension  $d$  is greater than or equal to 3. We start with an estimate on the normal derivative for a mixed boundary problem. We consider standard reflecting Brownian motion  $(\mathbb{Q}^x, X_t)$  in  $D$ , and we kill this process on hitting  $K$ . Fix a point  $x_0 \in D - K$  and choose  $\rho'$  small enough so that  $\text{dist}(x_0, \partial(D - K)) > 4\rho'$ .

**Proposition 3.1.** *Suppose  $D$  satisfies (2.2) and in addition  $\varphi$  is  $C^\infty$ . Suppose  $G$  consists of  $m$  components such that if  $A = \varphi(G)$ , then  $\sigma(\overline{A} - A) = 0$ . Let  $g(\cdot)$  be the Green function for  $X_t$  killed on hitting  $K \cup A$  with pole at  $x_0$ . Then  $(\partial g / \partial \nu)(y)$  exists at almost every point of  $\partial D$  (with respect to  $\sigma$ ) and there exists  $c_1$  such that*

$$\int_A \left( \frac{\partial g}{\partial \nu}(y) \right)^2 \sigma(dy) \leq c_1.$$

$c_1$  depends on  $\rho, \rho', \sup \varphi, \inf \varphi$ , and the Lipschitz constant of  $\varphi$  but does not otherwise depend on  $A$ . In particular,  $c_1$  does not depend on  $m$ .

**Proof.** The function  $g$  is harmonic in  $D - K - \{x_0\}$ . By standard results from PDE on the solution to the Dirichlet problem (see [GT], Sections 6.3 and 6.4),  $g$  can be extended to be  $C^\infty$  at every point of  $A$ ; this means that every point in  $A$  has a neighborhood in whose intersection with  $\overline{D}$  the function  $g$  is  $C^\infty$ . By standard results on the solution to the Neumann problem (see [GT], Section 6.7),  $g$  has a smooth extension up to the boundary in a neighborhood of each point in  $\partial D - \overline{A}$ . We make no claims at points in  $\overline{A} - A$ , but this set has surface measure 0.

Let us make the following assumptions about  $G$  and  $D$ . We will show they can be removed at the end of the proof. First we assume that each of the components of  $G$  has a piecewise smooth boundary (considered as a subset of the sphere  $\partial B(0, 1)$ ).

Let  $M$  be the Lipschitz constant of  $\varphi$ . Let  $\theta_1 = (0, \dots, 0, -1)$ . Choose  $r$  small (depending only on  $M, \sup \varphi$ , and  $\inf \varphi$ ) such that there exists a Lipschitz function  $\Gamma : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  whose Lipschitz constant is less than  $2M$  and with the property that the intersection of  $B(\varphi(\theta_1), r)$  with the region above the graph of  $\Gamma$  is the same as  $D \cap B(\varphi(\theta_1), r)$ . Our second assumption is that  $G \subseteq \partial B(0, 1) \cap B(\theta_1, r)$ .

If  $H$  is a  $(d-1)$ -dimensional hyperplane, let  $H^+$  be the half space that contains  $(\tilde{0}, y)$  for all  $y$  sufficiently large. We want to be able to apply Green's identities in  $D - K - B(x_0, \rho)$  with the function  $g$ , so to do so, we make the following assumption on  $D$  for now:

- (3.1) *For each component  $E_i$  of  $G$ , there exists  $\varepsilon_i > 0$  and a hyperplane  $H_i$  such that  $\{\varphi(\theta) \in \partial D : \text{dist}(\theta, E_i) \in (0, \varepsilon_i)\}$  is contained in  $H_i$  and  $\varphi(\theta)$  lies in  $H_i^+$  if  $\text{dist}(\theta, E_i^c) \in (0, \varepsilon_i)$ .*

Consider the domain  $C_i = \{(r, \theta) \in D - K : \text{dist}(\theta, \partial E_i) < \varepsilon_i/2\}$ . If we let  $C_i^R$  be the reflection of  $C_i \cap H_i^+$  across  $H_i$  and let  $C_i^*$  be the interior of  $C_i \cap H_i^+ \cup \overline{C_i^R}$ , then  $C_i^*$  has Lipschitz boundaries. By the reflection principle,  $g$  may be extended across  $H_i$ . By Dahlberg's theorem ([B], Section III.5),  $\partial g / \partial \nu$  is in  $L^2$  with respect to surface measure on  $\partial C_i^*$ , from which it follows that  $\partial g / \partial \nu$  is in  $L^2$  with respect to surface measure on  $\partial D \cap \{\varphi(\theta) : \text{dist}(\theta, E_i^c) \in (0, \varepsilon_i/2)\}$ .

We do this for each component, and conclude that we may apply Green's identities to the function  $g$  in  $D - K - B(x_0, 2\rho')$ . We can therefore conclude that  $\partial g/\partial\nu$  is the density of harmonic measure on  $D - K$  started at  $x_0$ ; this may be proved exactly as in [B], pp. 217-218.

Let  $S = \partial D \cap \overline{B(\varphi(\theta_1), r)^c}$ . Let  $F = D - B(0, 2\rho) - B(x_0, 2\rho')$  and let  $y_0 \in B(x_0, 4\rho') - B(x_0, 2\rho')$ . Since  $g$  is bounded and harmonic in  $F$ , then  $|\nabla g|$  is bounded on  $\partial B(0, 2\rho)$  and on  $\partial B(x_0, 2\rho')$ . By the PDE results mentioned in the first paragraph,  $|\nabla g|$  is also bounded in a neighborhood of points of  $S$ . Let  $(\mathbb{Q}^x, X_t)$  be the reflecting Brownian motion constructed in [BH1] and discussed in Section 2. Since  $\partial g/\partial y$  is a harmonic function, we have by Doob's optional stopping theorem

$$(3.2) \quad \frac{\partial g}{\partial y}(y_0) = \mathbb{E}^{y_0} \frac{\partial g}{\partial y}(X_{T(\partial F)}).$$

By the fact that  $\Gamma$  is a Lipschitz curve (with Lipschitz constant  $2M$ ) there exists  $c_2$  depending only on  $M$  such that the ratio of  $\partial g/\partial\nu$  to  $\partial g/\partial y$  is bounded above by  $c_2$  for  $y \in A$ . We have by the definition of harmonic measure that

$$(3.3) \quad \int_A \left( \frac{\partial g}{\partial \nu}(z) \right)^2 \sigma(dz) = \mathbb{E}^{x_0} \left[ \frac{\partial g}{\partial \nu}(X_{T(\partial F)}); X_{T(\partial F)} \in A \right].$$

Since  $g \geq 0$  in  $D$  and  $g = 0$  in  $A$ , then  $\partial g/\partial y \geq 0$  in  $A$ . The function

$$z \mapsto \mathbb{E}^z \left[ \left( \frac{\partial g}{\partial \nu} 1_A \right)(X_{T(\partial F)}) \right]$$

is harmonic, so by Harnack's inequality, there exists a  $c_3$  such that

$$(3.4) \quad \mathbb{E}^{x_0} \left[ \frac{\partial g}{\partial \nu}(X_{T(\partial F)}); X_{T(\partial F)} \in A \right] \leq c_3 \mathbb{E}^{y_0} \left[ \frac{\partial g}{\partial \nu}(X_{T(\partial F)}); X_{T(\partial F)} \in A \right].$$

Combining (3.2)-(3.4),

$$\begin{aligned} \int_A \left( \frac{\partial g}{\partial \nu}(z) \right)^2 \sigma(dz) &\leq c_3 \mathbb{E}^{y_0} \left[ \frac{\partial g}{\partial \nu}(X_{T(\partial F)}); X_{T(\partial F)} \in A \right] \\ &\leq c_2 c_3 \mathbb{E}^{y_0} \left[ \frac{\partial g}{\partial y}(X_{T(\partial F)}); X_{T(\partial F)} \in A \right] \\ &= c_4 \left( \frac{\partial g}{\partial y}(y_0) - \mathbb{E}^{y_0} \left[ \frac{\partial g}{\partial y}(X_{T(\partial F)}); X_{T(\partial F)} \in \partial B(x_0, 2\rho') \cup \partial B(0, 2\rho) \cup S \right] \right. \\ &\quad \left. - \mathbb{E}^{y_0} \left[ \frac{\partial g}{\partial y}(X_{T(\partial F)}); X_{T(\partial F)} \in \partial D - S - A \right] \right). \end{aligned}$$

As we argued above,  $\partial g/\partial y \geq 0$  a.e. in  $\partial D - S$ , while the first two terms on the right are bounded by constants depending only on  $\rho, \rho'$ , and  $M$ . Therefore

$$(3.5) \quad \int_A \left( \frac{\partial g}{\partial \nu}(z) \right)^2 \sigma(dz) \leq c_5.$$

We now show how to eliminate the assumptions made near the beginning of the proof. Suppose that we no longer assume  $G \subseteq B(\varphi(\theta_1), r)$ . Let  $A_0 = \varphi(G \cap B(\theta_1, r))$ , let  $g$  be the Green function for reflecting Brownian motion killed on hitting  $K \cup A$  with pole at  $x_0$ , and let  $g_0$  be the Green function for reflecting Brownian motion killed on hitting  $K \cup A_0$  with pole at  $x_0$ . Clearly

$$\mathbb{Q}^{x_0}(X_{T(K \cup A)} \in dy) \leq \mathbb{Q}^{x_0}(X_{T(K \cup A_0)} \in dy)$$

for  $y \in A_0$ , so on the set  $A_0$  the density of harmonic measure for reflecting Brownian motion killed on hitting  $K \cup A$ , which is  $\partial g / \partial \nu$ , is less than or equal to the density of harmonic measure for reflecting Brownian motion killed on hitting  $K \cup A_0$ , which is  $\partial g_0 / \partial \nu$ . By this fact and (3.5) applied to  $g_0$ ,

$$(3.6) \quad \int_{A \cap B(\varphi(\theta_1), r)} \left( \frac{\partial g}{\partial \nu}(z) \right)^2 \sigma(dz) \leq c_6.$$

By a rotation of the coordinate system, (3.6) holds when  $\theta_1$  is replaced by any other point of  $\partial B(0, 1)$ . Since  $\partial D$  can be covered by finitely many balls of the form  $B(\varphi(\theta), r)$  with  $\theta \in \partial B(0, 1)$ , summing gives

$$(3.7) \quad \int_A \left( \frac{\partial g}{\partial \nu}(z) \right)^2 \sigma(dz) \leq c_6.$$

Recall the  $\varphi$  is  $C^\infty$ . If the components  $E_i$  of  $G$  are each of the form  $Q \cap \partial B(0, 1)$ , where  $Q$  is a cube of side length less than  $h$ , we can achieve (3.1) by modifying  $\varphi$  (and hence  $D$ ) slightly. The smaller  $h$  is, the less we need to modify  $\varphi$ . Furthermore, we can approximate  $G$  as closely as we like by the union of such components. We can thus find a sequence of star-like domains  $D_m$  given by functions  $\varphi_m$  converging to  $D$  such that (3.7) holds when  $A$  is replaced by  $A_m = \varphi_m(G)$  and  $g$  is replaced by the Green function for reflecting Brownian motion on  $D_m$  and  $c_6$  is independent of  $m$ . By the limit argument of [B], pp. 217-218, we thus get (3.7) without any additional assumptions on  $G$ .  $\square$

**Corollary 3.2.** *Let  $D, G$  and  $A$  be as in Proposition 3.1. Let  $H$  be  $C^\infty$  with support in  $D - K$  and let*

$$u(x) = \mathbb{E}^x \int_0^{T_A \wedge T_K} H(X_s) ds.$$

*Then  $\partial u / \partial \nu$  exists a.e. on  $\partial D$  and there exists  $c_1$  depending only on  $\rho, \rho', \sup \varphi, \inf \varphi$ , and the Lipschitz constant of  $\varphi$  such that*

$$\int_A \left( \frac{\partial u}{\partial \nu}(y) \right)^2 \sigma(dy) \leq c_1.$$

**Proof.** The almost everywhere existence of  $\partial u/\partial \nu$  follows by the same PDE results as were used in the first paragraph of the proof of Proposition 3.1. If  $g(x, y)$  is the Green function for reflecting Brownian motion in  $D$  killed on hitting  $A \cup K$  with pole at  $x$ , then  $u(y) = \int g(x, y)H(x) dx$ . Since the support of  $H$  is a positive distance from  $\partial(D - K)$ , the result now follows from Proposition 3.1, Fubini's theorem, and Fatou's lemma.  $\square$

We next need an estimate that is essentially that of [JK], Section 4. Suppose  $D$  is a star-like Lipschitz domain:  $D = \{(r, \theta) : r < \varphi(\theta)\}$ , where  $\varphi$  is a positive Lipschitz function. Let us suppose  $\varphi$  is also  $C^\infty$ . Let  $\psi(\theta)$  be another  $C^\infty$  positive function that is strictly less than  $\varphi$  for all  $\theta$  and let  $E = \{(r, \theta) : r < \psi(\theta)\}$ . Let  $\delta < \text{dist}(\partial D, \partial E)/4$  and let  $E^\delta = \{x : \text{dist}(x, \partial E) < \delta\}$ . Recall that  $\nu_o$  is the outward pointing unit normal vector and  $\nu_o = -\nu$ .

**Proposition 3.3.** *Let  $D$  and  $E$  be as above, suppose  $f \in L^2(\partial D)$ , and suppose  $u$  is harmonic in  $(D - E) \cup E^\delta$  with  $\partial u/\partial \nu_o = f$  on  $\partial D$  and  $\int_{\partial D} u(z) \sigma(dz) = \int_{\partial E} u(z) \sigma(dz)$ . There exists  $c_1$  depending only on  $\delta, \sup \varphi, \inf \varphi$ , and the Lipschitz constants of  $\varphi$  and  $\psi$  such that*

$$\|N(\nabla u)\|_{2, \partial D} \leq c_1 \|f\|_{2, \partial D} + c_1 \sup_{E^\delta} |u|.$$

**Proof.** The proof follows [JK], Section 4, closely. First let us suppose  $f$  is smooth. Let  $h = \partial u/\partial \nu_o$  on  $\partial E$ , where by  $\nu_o$  on  $\partial E$  we mean the outward normal vector with respect to the domain  $D - E$ . Then clearly  $u$  is the solution to the Neumann problem in  $D - E$  with boundary functions  $f$  on  $\partial D$  and  $h$  on  $\partial E$ ; by Green's identity,  $\int_{\partial D} f = \int_{\partial E} h$ . Hence by [GT], Chapter 6,  $u$  is smooth on  $\overline{D - E}$ . Let  $x$  be the vector from the point 0 to the point  $x$ . If

$$R(x) = |\nabla u(x)|^2 x - 2(x \cdot \nabla u(x)) \nabla u(x) - (d - 2)u(x) \nabla u(x),$$

a calculation shows that  $\text{div} R(x) = 0$  in  $D - E$  since  $u$  is harmonic there. So by the divergence theorem,

$$(3.8) \quad \int_{\partial D} (R \cdot \nu_o)(z) \sigma(dz) = \int_{\partial E} (R \cdot \nu_o)(z) \sigma(dz).$$

Let us let  $K = \sup_{x \in E^\delta} |u(x)|$ . Since  $u$  is harmonic in  $E^\delta$ , then  $\nabla u$  is bounded by  $c_2 K$  there, and so the right hand side of (3.8) is bounded by  $c_3 K^2$ .

Let  $a(x) = x - (x \cdot \nu_o(x)) \nu_o(x)$ , so that

$$(x \cdot \nabla u) = (a(x) \cdot \nabla u) + (x \cdot \nu_o) \frac{\partial u}{\partial \nu_o}.$$

Let  $\nabla_t u$  denote the tangential component of  $\nabla u$ , that is,

$$\nabla_t u(x) = (\nabla u(x) \cdot v_1(x), \dots, \nabla u(x) \cdot v_{d-1}(x)),$$

where  $(v_1(x), \dots, v_{d-1}(x), \nu_o(x))$  forms an orthonormal set of vectors at  $x \in \partial(D - E)$ .

Then

$$|\nabla u|^2 = |\nabla_t u|^2 + \left(\frac{\partial u}{\partial \nu_o}\right)^2.$$

Since  $\partial u / \partial \nu_o = f$  on  $\partial D$ , we have

$$\int_{\partial D} |\nabla_t u|^2 (x \cdot \nu_o(x)) \sigma(dx) \leq \int_{\partial D} |f^2(x \cdot \nu_o)| + 2 \int_{\partial D} |(a \cdot \nabla u) f| + (d-2) \int_{\partial D} |u f| + c_3 K^2.$$

The domain is bounded, so  $x \cdot \nu_o$  and  $a$  are bounded, and because  $D$  is star-like, there exists  $c_4$  such that  $x \cdot \nu_o \geq c_4$  on  $\partial D$ . Hence

$$(3.9) \quad \int_{\partial D} |\nabla_t u|^2 \leq c_5 \left[ \int_{\partial D} f^2 + \int_{\partial D} |\nabla_t u| |f| + \int_{\partial D} |u| |f| + K^2 \right].$$

We said that  $\int_{\partial D} u = \int_{\partial E} u$  and  $|u| \leq K$  on  $\partial E$ . By the Poincaré inequality [M],

$$\begin{aligned} \frac{1}{\sigma(\partial D)} \int_{\partial D} u^2 &= \frac{1}{\sigma(\partial D)} \int_{\partial D} \left( u - \frac{1}{\sigma(\partial D)} \int_{\partial D} u \right)^2 + \left( \frac{1}{\sigma(\partial D)} \int_{\partial D} u \right)^2 \\ &\leq c_6 \left[ \int_{\partial D} |\nabla_t u|^2 + K^2 \right]. \end{aligned}$$

If we write  $F$  for  $\int_{\partial D} f^2$  and  $I$  for  $\int_{\partial D} |\nabla_t u|^2$ , then by the Cauchy-Schwarz inequality we have

$$\int_{\partial D} |u f| \leq c_7 (I + K^2)^{1/2} F^{1/2}.$$

Substituting in (3.9),

$$I \leq c_8 \left[ F + I^{1/2} F^{1/2} + K F^{1/2} + K^2 \right].$$

This implies there exists  $c_9$  depending only on  $c_8$  such that  $I \leq c_9 [K^2 + F]$ . Since  $\int_{\partial D} |\nabla u|^2 = I + F$ , we get  $\|\nabla u\|_{2, \partial D} \leq c_{10} [K^2 + F]$ . The result now follows for smooth  $f$  since the nontangential maximal function is bounded in  $L^2$  norm by the  $L^2$  norm of the function on the boundary (see [B], Section III.4, or [JK]). Finally we remove the restriction that  $f$  be smooth exactly as in [JK], pp. 39-42.  $\square$

Suppose  $D$  is a domain satisfying the hypotheses of Proposition 3.1, except that now we only assume  $\varphi$  is Lipschitz, not necessarily  $C^\infty$ . Let  $K$  be as above. Let  $H$  be a nonnegative  $C^\infty$  function with support in  $D - K$ ; let  $E$  be a smooth domain whose closure is contained in  $D$ , which contains the support of  $H$ , and which is star-like with respect to 0. Let  $G$  be an open subset of  $\partial B(0, 1)$  consisting of finitely many components such that if  $A = \varphi(G)$ , then  $\sigma(\overline{A} - A) = 0$ .

**Proposition 3.4.** *Let  $D, K, E, G, A$ , and  $H$  be as above. There exists a function  $u$  that is nonnegative and bounded,  $-(1/2)\Delta u = H$  in  $D - K$ ,  $\partial u/\partial\nu_o$  exists a.e.,  $u = 0$  a.e. on  $A$ ,  $\partial u/\partial\nu_o = 0$  a.e. on  $\partial D - A$ , and  $\|N(\nabla u)\|_{2,\partial D} < \infty$ .*

**Proof.** Let  $D_n = \{(r, \theta) : r < \varphi_n(\theta)\}$  be domains that are star-like with respect to 0, where the  $\varphi_n$  are  $C^\infty$  and that decrease to  $D$ ; we suppose also that  $\sup_n \|\nabla\varphi_n\|_\infty$  is finite.

Let  $A_n = \varphi_n(G)$ . Let  $(\mathbb{Q}_n^x, X_t)$  be standard reflecting Brownian motion in  $D_n$ , let expectation with respect to  $\mathbb{Q}_n^x$  be written  $\mathbb{E}_n^x$ , and let

$$u_n(x) = \mathbb{E}_n^x \int_0^{T_{A_n} \wedge T_K} H(X_s) ds.$$

Since the support of  $H$  is a compact subset of  $D$  and  $D$  is open,  $u_n$  is harmonic in a neighborhood of  $\partial D_n$ . By the discussion in Section 2,  $\partial u_n/\partial\nu = 0$  a.e. on  $(\partial D_n) - A_n$ . By Corollary 3.2,  $\partial u_n/\partial\nu$  exists a.e. on  $\partial D_n$  with  $L^2(\partial D_n)$  norm not depending on  $n$ . So by Proposition 3.3,  $N(\nabla u_n)$  is in  $L^2(\partial D_n)$  with a norm not depending on  $n$ .

We will show that a subsequence of the  $u_n$  converges to a function  $u$  that satisfies  $-(1/2)\Delta u = H$  in  $D$ ,  $u$  is nonnegative and bounded,  $u = 0$  a.e. on  $A$ ,  $\partial u/\partial\nu_o = 0$  a.e. on  $\partial D - A$ , and  $\|N(\nabla u)\|_{2,\partial D} < \infty$ .

Note each  $u_n(x)$  is nonnegative and

$$\|u_n\|_\infty \leq \|H\|_\infty \sup_{n,y} \mathbb{E}_n^y T_K.$$

By (2.4), the right hand side is finite. By the strong Markov property,

$$u_n(x) = \mathbb{E}_n^x \int_0^{T^{(\partial E)} \wedge T_K} H(X_s) ds + \mathbb{E}_n^x u_n(X_{T^{(\partial E)} \wedge T_K}).$$

The second term is harmonic inside  $E - K$  and the first is uniformly smooth inside  $E - K$  by standard results from PDE (see [GT], Section 6.4), since  $E$  is smooth. So the  $u_n$  are equicontinuous inside  $E - K$ . On the other hand, each  $u_n$  is harmonic outside the support of  $H$ . Therefore the  $u_n$  are equicontinuous on compact subsets of  $D$  and are uniformly bounded; hence there exists a subsequence which converges uniformly on compact subsets of  $D$ , say to  $u$ . By relabeling the  $D_n$ , let us suppose that the original sequence  $u_n$  converges.

Observe that if  $\theta \in \partial B(0, 1)$ , then  $V_\beta^D(\varphi(\theta)) \subseteq V_\beta^{D_n}(\varphi_n(\theta))$ , so  $N^D(\nabla u_n)(\varphi(\theta)) \leq N^{D_n}(\nabla u_n)(\varphi_n(\theta))$ . Since  $\nabla u_n$  converges uniformly to  $\nabla u$  in compact subsets of  $D - E$ , it follows easily by Fatou's lemma (cf. [JK], Section 4) that for  $\delta < \text{dist}(\partial D, \partial E)$ , we have  $\|N_\delta^D(\nabla u)\|_{2,\partial D} \leq c_1$ , where  $c_1$  depends only on  $\delta$  and the Lipschitz constant of  $\varphi$  but not on  $G$ . It is easy to see that  $\nabla u$  is bounded in  $D - U_\delta$  since  $H$  is smooth and  $u$  is the uniform limit of the  $u_n$  there, so we have

$$(3.10) \quad \|N^D(\nabla u)\|_{2,\partial D} \leq c_2.$$

Since  $u$  is bounded, it has nontangential limits a.e. We show the nontangential limit is 0 a.e. on  $A$ . Let  $v_n(x) = \mathbb{Q}_n^x(X_t \text{ hits } E \text{ before } A_n)$ . This is a harmonic function in  $D_n - E$  which equals 1 on  $E$  and has nontangential limits 0 on  $A_n$ . Let  $\varepsilon > 0$  and  $\theta \in G$ . Since  $v_n$  is bounded by 1 and is 0 on  $A_n$ , by Lemma 2.1 there exists  $\delta$  such that if  $x \in V_\beta^{D_n}(\varphi_n(\theta))$  and  $|x - \varphi_n(\theta)| < \delta$ , then  $v_n(x) < \varepsilon$ . So if  $x \in V_\beta^D(\varphi(\theta)) \subseteq V_\beta^{D_n}(\varphi_n(\theta))$  and  $|x - \varphi(\theta)| < \delta/2$ , then for  $n$  large,  $|x - \varphi_n(\theta)| < \delta$ . Since  $u_n$  has nontangential limits 0 on  $A_n$  and is harmonic in  $D_n - E$ ,

$$0 \leq u_n(x) \leq \sup_n \|u_n\|_\infty \mathbb{Q}_n^x(T_E < T_{A_n}) \leq v_n(x) \sup_n \|u_n\|_\infty.$$

Hence  $u(x) \leq \varepsilon \sup_n \|u_n\|_\infty$ . This shows  $u(x) \rightarrow 0$  as  $x \rightarrow \varphi(\theta)$  nontangentially.

Since  $\|N_D(\nabla u)\|_{2,\partial D} \leq c_2$ , then  $\nabla u$  converges nontangentially a.e. and so  $\partial u/\partial \nu_o$  exists a.e. It remains to show that  $\partial u/\partial \nu_o = 0$  a.e. on  $\partial D - A$ . Let  $h(\theta)$  be a smooth function with support in  $G^c$ . Let  $f_n(x) = \mathbb{E}_n^x h(X_{T(\partial D_n)})$ ,  $f(x) = \mathbb{E}^x h(X_{T(\partial D)})$ , where  $(\mathbb{Q}^x, X_t)$  is reflecting Brownian motion on  $D$ . Note the restriction of  $f_n$  to  $\partial D_n$  is supported on  $A_n^c$  and the restriction of  $f$  to  $\partial D$  is supported on  $A^c$ .

Let  $(\mathbb{P}^x, W_t)$  be standard Brownian motion on  $\mathbb{R}^d$ . Up until times  $T(\partial D_n)$  and  $T(\partial D)$ ,  $(\mathbb{Q}_n^x, X_t)$  and  $(\mathbb{Q}^x, X_t)$ , respectively, have the same law as  $(\mathbb{P}^x, W_t)$ . So  $f_n(x) = \mathbb{E}_{\mathbb{P}^x} h(W_{T(\partial D_n)})$  and  $f(x) = \mathbb{E}_{\mathbb{P}^x} h(W_{T(\partial D)})$ . Since  $T_{\partial D_n} \downarrow T_{\partial D}$  and  $h$  is smooth, it follows that  $f_n$  converges to  $f$  on  $D$ . Since  $f_n$  and  $f$  are harmonic, the convergence is uniform on compact subsets of  $D$ .

By Green's first identity, since  $f_n$  and  $f$  are harmonic in  $D_n - E$  and  $D - E$ , respectively, and  $\partial u_n/\partial \nu_o = 0$  on  $(\partial D_n) - A_n$ ,

$$\int_{D_n - E} \nabla f_n \cdot \nabla u_n = \int_{\partial D_n} f_n \frac{\partial u_n}{\partial \nu_o} + \int_{\partial E} f_n \frac{\partial u_n}{\partial \nu_o} = \int_{\partial E} f_n \frac{\partial u_n}{\partial \nu_o}$$

and

$$\int_{D - E} \nabla f \cdot \nabla u = \int_{\partial D} f \frac{\partial u}{\partial \nu_o} + \int_{\partial E} f \frac{\partial u}{\partial \nu_o}.$$

(Recall  $\nu_o$  is the outward normal vector for the domains  $D_n - E$  or  $D - E$ .)

We will show

$$(3.11) \quad \int_{D_n - E} \nabla f_n \cdot \nabla u_n \rightarrow \int_{D - E} \nabla f \cdot \nabla u.$$

Since  $f_n \rightarrow f$  and  $u_n$  is harmonic on  $\partial E$  and uniformly bounded in  $n$  in a neighborhood of  $E$ ,  $\partial u_n/\partial \nu_o \rightarrow \partial u/\partial \nu_o$  on  $E$ . So if (3.11) holds,  $\int_{\partial D} f(\partial u/\partial \nu_o) = 0$ . If this holds for all such  $h$ , then  $\partial u/\partial \nu_o = 0$  a.e. on  $A^c$ . So it remains to show (3.11).

Recall the definition of  $U_r$  from (2.1). We have  $\nabla f_n \rightarrow \nabla f$  uniformly on  $D - U_r$  and  $\nabla u_n \rightarrow \nabla u$  uniformly on  $D - U_r$ , so

$$(3.12) \quad \int_{D - U_r} \nabla f_n \cdot \nabla u_n \rightarrow \int_{D - U_r} \nabla f \cdot \nabla u.$$



Since  $h$  is smooth, by [JK], Theorem 4.13, there exists  $c_3$  independent of  $n$  such that

$$\int_{\partial D_n} (N(\nabla f_n)(y))^2 \sigma(dy) \leq c_3$$

and

$$\int_{\partial D} (N(\nabla f)(y))^2 \sigma(dy) \leq c_3.$$

Let  $\varepsilon > 0$ . If  $r < \varphi_n(\theta)$ , then  $|\nabla f_n(r, \theta)| \leq N(\nabla f_n)(\varphi_n(\theta))$ . So  $\int_{(D_n-D) \cup U_r} |\nabla f_n|^2$  can be made less than  $\varepsilon$  if  $r$  is small enough and  $n$  is large enough, and similarly  $\int_{U_r} |\nabla f|^2 < \varepsilon$  if  $r$  is small enough. We have

$$\begin{aligned} \int_{(D_n-D) \cup U_r} |\nabla f_n \cdot \nabla u_n| &\leq \left( \int_{(D_n-D) \cup U_r} |\nabla f_n|^2 \right)^{1/2} \left( \int_{(D_n-D) \cup U_r} |\nabla u_n|^2 \right)^{1/2} \\ &\leq c_4 \sup_n \varepsilon \|N(\nabla u_n)\|_{2, \partial D} \end{aligned}$$

and

$$\int_{U_r} |\nabla f \cdot \nabla u| \leq \left( \int_{U_r} |\nabla f|^2 \right)^{1/2} \left( \int_{U_r} |\nabla u|^2 \right)^{1/2} \leq c_4 \varepsilon \|N(\nabla u)\|_{2, \partial D}.$$

Combining with (3.12) and using the fact that  $\varepsilon$  is arbitrary gives (3.11).  $\square$

**Proposition 3.5.** *Let  $D, K, E$ , and  $H$  be as in Proposition 3.4. Let  $G$  be an arbitrary open set in  $\partial B(0, 1)$  and  $A = \varphi(G)$ . Then there exists a function  $u(x)$  satisfying the conclusions of Proposition 3.4.*

**Proof.** Let  $G_m$  be open sets satisfying the hypotheses of Proposition 3.1 and increasing to an open set  $G$  and let  $A_m = \varphi(G_m)$  and  $A = \varphi(G)$ . Let  $u_m(x)$  be the corresponding functions given by Proposition 3.4. The  $u_m$  are uniformly bounded, harmonic in  $D - E$ , and satisfy  $-(1/2)\Delta u_m = H$  in  $E - K$ . As  $m$  increases,  $A_m \uparrow A$ . Using the notation of the proof of Proposition 3.4, observe that as  $m$  increases,  $T_{A_m}$  decreases, and so  $\mathbb{E}_n^x \int_0^{T_{A_m} \wedge T_K} H(X_s) ds$  decreases. It follows that  $u_m(x)$  decreases as  $m$  increases for each  $x$ . Let  $u(x) = \lim_{m \rightarrow \infty} u_m(x)$ . By the harmonicity and boundedness of the  $u_m$ , the convergence is uniform on compact subsets of  $D - E$ . Therefore  $u$  is harmonic in  $D - E$ , bounded in  $D$ , 0 on  $K$ , and  $-(1/2)\Delta u = H$  in  $E - K$ .

Suppose  $x \in V_\beta(z)$  for some  $z \in A$ . Then  $z \in A_m$  for some  $m$  and given  $\varepsilon$ , there exists  $\delta$  such that if  $|x - z| < \delta$ , then  $0 \leq u_m(x) < \varepsilon$ . Therefore  $u(x) \leq u_m(x) < \varepsilon$ . This shows that  $u$  has nontangential limits 0 a.e. on  $A$ .

By Fatou's lemma and the corresponding result for the  $u_m$ ,  $\|N(\nabla u)\|_{2, \partial D} \leq c_1$ . So  $\partial u / \partial \nu_o$  exists a.e., and we must show that it is 0 a.e. on  $A^c$ . Suppose there exists a set  $B$  of positive surface measure contained in  $A^c$  on which  $\partial u / \partial \nu_o > r$  for some  $r > 0$ . (The

case where  $\partial u/\partial\nu_o$  is negative is treated similarly.) Pick  $f$  smooth on  $\partial D$  such that  $f = 1$  on  $B$  and  $\int_{\partial D} f(\partial u/\partial\nu_o) > r\sigma(B)/2$ . We also require

$$\left(\int_{\partial D-B} f^2\right)^{1/2} < \frac{r\sigma(B)}{4 \sup_m \|N(\nabla u_m)\|_{2,\partial D}}.$$

We can find such a  $f$  by taking smooth  $f$  decreasing to  $1_B$ . Extend  $f$  to  $D$  by defining  $f(x) = \mathbb{E}^x f(X_{T(\partial D)})$ , where  $(\mathbb{Q}^x, X_t)$  is reflecting Brownian motion on  $D$ . Since  $B \subseteq A^c \subseteq A_m^c$ ,  $\partial u_m/\partial\nu_o = 0$  on  $B$  and

$$\left|\int_{\partial D} f \frac{\partial u_m}{\partial\nu_o}\right| = \left|\int_{\partial D-B} f \frac{\partial u_m}{\partial\nu_o}\right| \leq \left(\int_{\partial D-B} f^2\right)^{1/2} \left(\int_{\partial D} \left|\frac{\partial u_m}{\partial\nu_o}\right|^2\right)^{1/2} < r\sigma(B)/4.$$

Now by Green's identity on  $D$  and the fact that  $f$  is harmonic in  $D$ ,

$$\int_{\partial D} f \frac{\partial u_m}{\partial\nu_o} = \int_D \nabla f \cdot \nabla u_m,$$

and similarly to Proposition 3.4 but easier, this converges to  $\int_D \nabla f \cdot \nabla u = \int_{\partial D} f(\partial u/\partial\nu_o)$ . This implies that

$$\frac{r\sigma(B)}{4} \geq \lim_m \int_{\partial D} f \frac{\partial u_m}{\partial\nu_o} = \int_{\partial D} f \frac{\partial u}{\partial\nu_o} > \frac{r\sigma(B)}{2},$$

a contradiction. Therefore there exists no such subset  $B$ . □

**Corollary 3.6.** *Let  $D, K, E, H, G$ , and  $A$  be as in Proposition 3.5. There exist reals  $r_n \uparrow 1$  and functions  $F_n : D \rightarrow \mathbb{R}$  that are  $C^\infty$ , the  $F_n$  are nonnegative and uniformly bounded,  $-(1/2)\Delta F_n \rightarrow H$  uniformly in  $E - K$ ,  $F_n = 0$  on  $B(0, \rho/r_n)$ ,  $F_n \rightarrow 0$  a.e. on  $A$ ,  $\partial F_n/\partial\nu_o \rightarrow 0$  a.e. on  $A^c$ , and  $\sup_n \|N(\nabla F_n)\|_{2,\partial D} < \infty$ .*

**Proof.** Let  $u$  be the function constructed in Proposition 3.5, let  $r_n \uparrow 1$  and let

$$F_n(x) = u(r_n x), \quad x \in D. \quad \square$$

#### 4. Uniqueness.

For now we suppose  $D$  satisfies (2.2) and the dimension  $d \geq 3$ . Let  $(\mathbb{Q}^x, X_t)$  denote a standard reflecting Brownian motion, let  $x_0 \in D$ , and let  $\mathbb{P}$  be a probability measure that is a solution to (2.10). Without loss of generality we may suppose  $x_0 \neq 0$ . Our main goal is to show  $\mathbb{P} = \mathbb{Q}^{x_0}$ . Let  $\rho < \min(|x_0|, \text{dist}(0, \partial D))/4$  and define  $K = B(0, \rho)$ . Let  $\theta_t$  be the usual shift operators.

The process  $L_t$  is a continuous process, so if  $M > 0$  and  $\xi_1(M) = \inf\{t : L_t \geq M\}$ , then  $\xi_1(M) > 0$ . Since  $L_{t \wedge \xi_1(M)}$  is the uniform limit of  $A_m(t \wedge \xi_1(M))$ , where  $A_m$  is defined by (2.6), it follows that  $\xi_2(M) = \inf\{t : \sup_m A_m(t \wedge \xi_1(M)) \geq 2M\}$  is also strictly positive, a.s. We let  $\xi_3 = \xi_3(M) = \xi_1(M) \wedge \xi_2(M) \wedge M$ , and observe that  $0 < \xi_3(M)$  and  $\xi_3(M) \rightarrow \infty$  as  $M \rightarrow \infty$ .

We define a new probability measure  $\mathbb{P}'$  that agrees with  $\mathbb{P}$  up to time  $\xi_3$  and agrees with  $\mathbb{Q}^{x_0}$  after time  $\xi_3$  as follows. If  $A \in \mathcal{F}_{\xi_3}$  and  $B \in \mathcal{F}_\infty$ , let

$$\mathbb{P}'(B \circ \theta_{\xi_3} \cap A) = \mathbb{E}_{\mathbb{P}}(\mathbb{Q}^{X(\xi_3)}(B); A).$$

This determines the probability measure  $\mathbb{P}'$  on  $\mathcal{F}_\infty$  ([SV2], Chapter 6).  $\mathbb{P}'$  is a solution to (2.10) up to time  $\xi_3$  since it agrees with  $\mathbb{P}$  on  $\mathcal{F}_{\xi_3}$ .  $\mathbb{P}'$  solves (2.10) shifted by an amount  $\xi_3$  by the fact that for each  $x$ ,  $\mathbb{Q}^x$  is a solution to (2.10) starting at  $x$ . If we show that  $\mathbb{P}' = \mathbb{Q}^{x_0}$ , then  $\mathbb{P} = \mathbb{Q}^{x_0}$  on  $\mathcal{F}_{\xi_3(M)}$ , and letting  $M \rightarrow \infty$ , we obtain  $\mathbb{P} = \mathbb{Q}^{x_0}$ .

The reason we work with  $\mathbb{P}'$  is the following.

**Proposition 4.1.** (a)  $\mathbb{E}_{\mathbb{P}'} L_{T_K} < \infty$ .

(b)  $\mathbb{E}_{\mathbb{P}'} \sup_m A_m(T_K) < \infty$ .

(c)  $\mathbb{E}_{\mathbb{P}'} T_K < \infty$ .

**Proof.** We have  $A_m(T_K) = A_m(\xi_3) + A_m(T_K) \circ \theta_{\xi_3}$  and letting  $m \rightarrow \infty$  we obtain  $L_{T_K} = L_{\xi_3} + L_{T_K} \circ \theta_{\xi_3}$ . So by the definition of  $\mathbb{P}'$ ,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}'} L_{T_K} &= \mathbb{E}_{\mathbb{P}} L_{\xi_3} + \mathbb{E}_{\mathbb{P}'}(L_{T_K} \circ \theta_{\xi_3}) \\ &= \mathbb{E}_{\mathbb{P}} L_{\xi_3} + \mathbb{E}_{\mathbb{P}}(\mathbb{E}^{X(\xi_3)} L_{T_K}). \end{aligned}$$

By the definition of  $\xi_3$ , the first term is bounded by  $M$ , while the second term is finite by the discussion following (2.4). The proof of assertion (c) is essentially the same.

Since

$$\sup_m A_m(T_K) \leq \sup_m A_m(\xi_3) + (\sup_m A_m(T_K)) \circ \theta_{\xi_3},$$

the proof of (b) is similar. □

We now drop the primes from  $\mathbb{P}'$ , and without loss of generality we may suppose that  $\mathbb{E}_{\mathbb{P}} L_{T_K} < \infty$ ,  $\mathbb{E}_{\mathbb{P}} \sup_m A_m(T_K) < \infty$ , and  $\mathbb{E}_{\mathbb{P}} T_K < \infty$ .

Define a measure  $\mu$  on  $D - K$  by

$$\mu(B) = \mathbb{E}_{\mathbb{P}} \int_0^{T_K} 1_B(X_s) ds,$$

the amount of time spent in  $B$  before hitting  $K$ .

**Proposition 4.2.**  $\mu$  is absolutely continuous with respect to Lebesgue measure. If we let  $h$  denote the Radon-Nikodym derivative, we may choose  $h$  to be finite, nonnegative, and harmonic in  $D - K - \{x_0\}$ .

**Proof.** The nonnegativity is clear. Let  $B_0 \subseteq B_1 \subseteq B_2$  be any three concentric balls contained in  $D - K - \{x_0\}$  such that  $\overline{B_0} \subseteq B_1$ ,  $\overline{B_1} \subseteq B_2$ , and  $\overline{B_2} \subseteq D - K - \{x_0\}$ . If we prove that  $\mu$  restricted to  $B_0$  has a Radon-Nikodym derivative with respect to Lebesgue measure that may be chosen to be harmonic in  $B_0$ , then since  $B_0$  is arbitrary we will obtain our result.

Let  $S_1 = \inf\{t : X_t \in B_1\}$ ,  $T_1 = \inf\{t > S_1 : X_t \notin B_2\}$ , and for  $i \geq 1$  let  $S_{i+1} = \inf\{t > T_i : X_t \in B_1\}$  and  $T_{i+1} = \inf\{t > S_{i+1} : X_t \notin B_2\}$ . If  $B \subseteq B_0$ , then

$$(4.1) \quad \mathbb{E}_{\mathbb{P}} \int_0^{T_K} 1_B(X_s) ds = \sum_{i=1}^{\infty} \mathbb{E}_{\mathbb{P}} \left[ \int_{S_i}^{T_i} 1_B(X_s) ds; S_i < T_K \right].$$

Note that under  $\mathbb{P}$ , the process  $\int_0^t 1_B(X_s) ds$  cannot increase before time  $S_1$  because  $x_0 \notin B_2$ ; since  $K \cap B_2 = \emptyset$ , if  $S_i < T_K$ , then  $T_i < T_K$ .

The law of  $X_s \circ \theta_{S_i}$  under a regular conditional probability for  $\mathbb{E}[\cdot | \mathcal{F}_{S_i}]$  is by Proposition 2.3 a solution to (2.10) started at  $X_{S_i}$ . Started at  $x \in D - K$ , the law of  $X_t$  is the same as that of a standard  $d$ -dimensional Brownian motion up to time  $T_{\partial D}$ . Therefore  $\mathbb{E}_{\mathbb{P}}[\int_{S_i}^{T_i} 1_B(X_s) ds | \mathcal{F}_{S_i}]$  is the same as the amount of time Brownian motion started at  $X_{S_i}$  spends in  $B$  up until leaving  $B_2$ . If  $g_{B_2}(x, y)$  is the Green function for standard  $d$ -dimensional Brownian motion killed on exiting  $B_2$ , we have then

$$(4.2) \quad \mathbb{E}_{\mathbb{P}} \left[ \int_{S_i}^{T_i} 1_B(X_s) ds | \mathcal{F}_{S_i} \right] = \int_B g_{B_2}(X_{S_i}, y) dy.$$

The law of  $X \circ \theta_{T_i}$  under a regular conditional probability for  $\mathbb{E}_{\mathbb{P}}[\cdot | \mathcal{F}_{T_i}]$  is a solution to (2.10) started at  $X_{T_i}$ . A solution started at  $X_{T_i}$  is a standard Brownian motion up until hitting  $\partial D$ . By the support theorem for Brownian motion ([B], p. 59), there exists  $\rho < 1$  such that

$$\mathbb{Q}^y(T_{B_1} < T_K) \leq \rho, \quad y \in \partial B_2.$$

Hence

$$\mathbb{Q}^{X(T_i)}(T_{B_1} < T_K) \leq \rho,$$

or

$$\mathbb{P}(S_{i+1} < T_K | \mathcal{F}_{T_i}) \leq \rho, \quad \text{a.s.}$$

From this we deduce

$$\begin{aligned} \mathbb{P}(S_{i+1} < T_K) &= \mathbb{P}(S_{i+1} < T_K, S_i < T_K) = \mathbb{E}_{\mathbb{P}}[\mathbb{P}(S_{i+1} < T_K | \mathcal{F}_{T_i}); S_i < T_K] \\ &\leq \rho \mathbb{P}(S_i < T_K). \end{aligned}$$

By induction,  $\mathbb{P}(S_i < T_K) \leq \rho^i$ .

Combining with (4.1) and (4.2) and using Fubini's theorem,

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_K} 1_B(X_s) ds = \int_B \sum_{i=1}^{\infty} \mathbb{E}_{\mathbb{P}}[g_{B_2}(X_{S_i}, y); S_i < T_K] dy.$$

$g_{B_2}(x, y)$  is harmonic in  $y \in B_0$  when  $x \in \partial B_1$ ; therefore  $\mathbb{E}_{\mathbb{P}}[g_{B_2}(X_{S_i}, y); S_i < T_K]$  is harmonic. Since  $g_{B_2}(x, y)$  is bounded over  $x \in \partial B_1, y \in B_0$ , then

$$(4.3) \quad \sum_{i=i_0}^{\infty} \mathbb{E}_{\mathbb{P}}[g_{B_2}(X_{S_i}, y); S_i < T_K] \leq \sum_{i=i_0}^{\infty} c_1 \mathbb{P}(S_i < T_K) \leq \sum_{i=i_0}^{\infty} c_1 \rho^i < \infty.$$

Let

$$h(y) = \sum_{i=1}^{\infty} \mathbb{E}_{\mathbb{P}}[g_{B_2}(X_{S_i}, y); S_i < T_K].$$

In view of (4.3) the sum converges uniformly over  $y \in B_0$  and hence  $h$  is finite and harmonic in  $B_0$ .  $\square$

Since  $h$  is nonnegative and harmonic in  $D - K - \{x_0\}$ , the nontangential maximal function of  $h$  is finite a.e. in a neighborhood of  $\partial D$ , i.e., for  $\varepsilon$  less than  $\rho'$ ,

$$N_{\varepsilon}(h)(z) < \infty, \quad \text{for almost every } z \in \partial D.$$

By (2.3), the Green function for  $D - K$  with pole at  $x_0$  is bounded in  $D - B(x_0, \rho')$ , say by  $R$ . We construct a sawtooth domain

$$D_0 = \bigcup \{V_{\beta}(z) : z \in \partial D, N_{\varepsilon}(h)(z) \leq 3R\}.$$

$D_0$  is a Lipschitz domain, and since it is contained in  $D$ , still star-like with respect to 0. Let  $A = \partial D_0 - \partial D$ .

**Lemma 4.3.** *There exists  $c_1$  depending only on  $\varepsilon$  and  $R$  such that*

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_K \wedge T_A} |\varphi(X_t)| dL_t \leq c_1 \int_{\partial D - A} |\varphi(y)| \sigma(dy).$$

**Proof.** First suppose  $\varphi$  is nonnegative and continuous on  $\partial D$  and extend  $\varphi$  to be nonnegative and continuous in  $D$  as well. Since  $L_t$  is the uniform limit of the  $A_m(t)$ ,

$$\int_0^{T_K \wedge T_A \wedge t} \varphi(X_s) dA_m(s) \rightarrow \int_0^{T_K \wedge T_A \wedge t} \varphi(X_s) dL_s.$$

By Proposition 4.1(b) and dominated convergence,

$$(4.4) \quad \mathbb{E}_{\mathbb{P}} \int_0^{T_K \wedge T_A \wedge t} \varphi(X_s) dL_s = \lim_{m \rightarrow \infty} \mathbb{E}_{\mathbb{P}} \int_0^{T_K \wedge T_A \wedge t} \varphi(X_s) dA_m(s).$$

$h$  is bounded by  $3R$  in a neighborhood of  $\partial D_0$ . Recall the definition of  $U_r$  from (2.1). Because  $h$  is harmonic in  $D - K - B(x_0, \rho') - U_r$ , it is bounded by a constant  $c_2$  there. Since  $dA_m(s) = f_m(X_s) ds$ , the right hand side in (4.4) is bounded by

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \int_0^{T_A \wedge T_K \wedge t} \varphi(X_s) f_m(X_s) ds &\leq \int 1_{D_0}(y) \varphi(y) f_m(y) h(y) dy \\ &\leq c_3 \int_D \varphi(y) f_m(y) dy, \end{aligned}$$

where  $c_3 = c_2 \vee R$ . Since  $f_m(y) dy$  converges weakly to  $\sigma(dy)$  and  $\varphi$  is continuous,

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_A \wedge T_K \wedge t} \varphi(X_s) dL_s \leq c_3 \int_{\partial D} \varphi(y) \sigma(dy).$$

Now let  $t \rightarrow \infty$ . By linearity and a limit argument,

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_A \wedge T_K} |\varphi(X_s)| dL_s \leq c_3 \int_{\partial D} |\varphi(y)| \sigma(dy)$$

for all  $\varphi$  bounded and measurable on  $\partial D$ . Finally, since  $L_t$  grows only when  $X_t \in \partial D$ ,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \int_0^{T_K \wedge T_A} |\varphi(X_s)| dL_s &= \mathbb{E}_{\mathbb{P}} \int_0^{T_A \wedge T_K} |(\varphi 1_{(\partial D - A)})(X_s)| dL_s \\ &\leq c_3 \int |(\varphi 1_{(\partial D - A)})(y)| \sigma(dy). \end{aligned}$$

This completes the proof.  $\square$

Let  $H$  be a  $C^\infty$  function with support in  $D - K - \{x_0\}$ . The key proposition is the following.

**Proposition 4.4.** *Let  $u(x) = \mathbb{E}^x \int_0^{T_A \wedge T_K} H(X_s) ds$ . Then if  $x_0 \in D - K$ ,*

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_A \wedge T_K} H(X_s) ds = u(x_0).$$

**Proof.** Let  $E$  be star-like, contained in  $D$ , and containing the support of  $H$ . Construct the  $F_n$  as in Corollary 3.6. For  $r > 1$  let  $K_r = B(0, r\rho)$ .  $F_n$  is  $C^2$  on  $D - K_r$ , so by Ito's formula,

$$(4.5) \quad \begin{aligned} F_n(X(T_{K_r} \wedge T_A \wedge t)) - F_n(X_0) &= \text{martingale} + \frac{1}{2} \int_0^{T_{K_r} \wedge T_A \wedge t} \Delta F_n(X_s) ds \\ &\quad + \int_0^{T_{K_r} \wedge T_A \wedge t} \frac{\partial F_n}{\partial \nu}(X_s) dL_s. \end{aligned}$$

Take expectations with respect to  $\mathbb{P}$ . By Lemma 4.3, the expectation of the local time term in (4.5) (i.e., the last term on the right) is bounded by

$$c_1 \int_{\partial D-A} \left| \frac{\partial F_n}{\partial \nu}(y) \right| \sigma(dy).$$

$F_n$  is bounded and  $\Delta F_n$  is bounded in  $D$ , so letting  $t \rightarrow \infty$ , we obtain

$$\mathbb{E}_{\mathbb{P}} F_n(X(T_{K_r} \wedge T_A)) - F_n(x_0) = \frac{1}{2} \mathbb{E}_{\mathbb{P}} \int_0^{T_{K_r} \wedge T_A} \Delta F_n(X_s) ds + R,$$

where

$$|R| \leq c_1 \int_{\partial D-A} \left| \frac{\partial F_n}{\partial \nu}(y) \right| \sigma(dy).$$

Next let  $n \rightarrow \infty$ . By Corollary 3.6  $\partial F_n / \partial \nu$  is in  $L^2(\sigma)$  with a bound independent of  $n$  and  $\partial F_n / \partial \nu \rightarrow 0$  a.e. on  $\partial D - A$ . We also have that  $(1/2)\Delta F_n \rightarrow -H$  uniformly on  $D - K_r$  and  $F_n \rightarrow u$  uniformly. So we obtain

$$\mathbb{E}_{\mathbb{P}} u(X(T_{K_r} \wedge T_A)) - u(x_0) = -\mathbb{E}_{\mathbb{P}} \int_0^{T_{K_r} \wedge T_A} H(X_s) ds.$$

The function  $u$  is 0 on  $A$  and on  $K$  and  $H$  is bounded. So by dominated convergence on the left and monotone convergence on the right, letting  $r \downarrow 1$ ,

$$-u(x_0) = -\mathbb{E}_{\mathbb{P}} \int_0^{T_K \wedge T_A} H(X_s) ds. \quad \square$$

**Corollary 4.5.** *If  $x_0 \in D - K$ ,*

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_A \wedge T_K} H(X_s) ds = \mathbb{E}^{x_0} \int_0^{T_A \wedge T_K} H(X_s) ds$$

*for all bounded functions  $H$ .*

**Proof.** This follows from Proposition 4.4 by a limit argument and the fact that the quantity  $\mathbb{E}_{\mathbb{P}} \int_0^{T_K \wedge T_A} 1_{\{x_0\}}(X_s) ds$  is 0 since  $X_t$  behaves like a Brownian motion in a neighborhood of  $x_0$ .  $\square$

**Corollary 4.6.** *If  $x_0 \in D - K$ ,*

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_K} H(X_s) ds = \mathbb{E}^{x_0} \int_0^{T_K} H(X_s) ds$$

for all bounded functions  $H$ .

**Proof.** Recall the definition of  $R$  following the proof of Proposition 4.2. If  $H$  has support in  $D - K - B(x_0, \rho')$ , then by the definition of  $R$ ,

$$\mathbb{E}^{x_0} \int_0^{T_K} H(X_s) 1_B(X_s) ds \leq R \|H\|_\infty |B|$$

for  $B \subseteq D$ . This and Corollary 4.5 imply that  $h \leq R$ , a.e., on  $D - B(x_0, \rho')$ . Since  $h$  is harmonic, it is continuous, or  $h \leq R$  on  $D - B(x_0, \rho')$ . This implies  $N_\varepsilon(h)(z) \leq R$ , so  $D_0 = D$  and hence  $A = \emptyset$ .  $\square$

We would like the conclusion of Corollary 4.6 to hold for all  $x_0$ , even for  $x_0 \in \partial D$ .

**Proposition 4.7.** *If  $x_0 \in \partial D$ ,*

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_K} H(X_s) ds = \mathbb{E}^{x_0} \int_0^{T_K} H(X_s) ds.$$

**Proof.** Let  $x_0 \in \partial D$ . Let  $\xi_4(n) = \inf\{t : |X_t - x_0| \geq 1/n\}$  and  $\xi_5(m) = \inf\{t : \text{dist}(X_t, \partial D) \geq 1/m\}$ . Choose  $m_n$  such that  $\mathbb{P}(\xi_5(m_n) > \xi_4(n)) < 1/n$ ; this is possible since starting at  $x_0$  the process under  $\mathbb{P}$  leaves  $\partial D$  immediately. Let  $\xi_6(n) = \xi_4(n) \wedge \xi_5(m_n) \wedge 1/n$ . As in Corollary 4.5 it suffices to prove the proposition for  $H$  in  $C^\infty$  with support in  $D - K$ . So for  $n$  sufficiently large,  $X_t$  will not be in the support of  $H$  when  $t \leq \xi_6(n)$  and

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_K} H(X_s) ds = \mathbb{E}_{\mathbb{P}} \int_{\xi_6(n)}^{T_K} H(X_s) ds.$$

The law of the process  $X_s \circ \theta_{\xi_6(n)}$  under a regular conditional probability for  $\mathbb{E}_{\mathbb{P}}[\cdot | \mathcal{F}_{\xi_6(n)}]$  is a solution to (2.10) started at  $X_{\xi_6(n)}$ . On the set where  $X_{\xi_6(n)} \notin \partial D$ , by Corollary 4.6 we have  $\mathbb{E}_{\mathbb{P}}[\int_0^{T_K} H(X_s) ds | \mathcal{F}_{\xi_6(n)}] = u(X_{\xi_6(n)})$ , where

$$(4.6) \quad u(x) = \mathbb{E}^x \int_0^{T_K} H(X_s) ds.$$

So

$$\begin{aligned} \left| \mathbb{E}_{\mathbb{P}} \int_{\xi_6(n)}^{T_K} H(X_s) ds - \mathbb{E}_{\mathbb{P}} u(X_{\xi_6(n)}) \right| &\leq (\|H\|_\infty \mathbb{E}_{\mathbb{P}} T_K + \|u\|_\infty) \mathbb{P}(X_{\xi_6(n)} \in \partial D) \\ &\leq c_1/n. \end{aligned}$$



By (2.7),  $u$  is continuous in  $B(x_0, 1/n) \cap \overline{D}$ , so letting  $n \rightarrow \infty$ , we have  $u(X_{\xi_6(n)}) \rightarrow u(x_0)$ , and hence

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_K} H(X_s) ds = \mathbb{E}_{\mathbb{P}} u(x_0) = u(x_0). \quad \square$$

**Proposition 4.8.** *Let  $S_\lambda H = \mathbb{E}_{\mathbb{P}} \int_0^{T_K} e^{-\lambda t} H(X_t) dt$ . Then for all  $x_0 \in \overline{D} - K$  and for all  $\lambda < 1/(2 \sup_y \mathbb{E}^y T_K)$ ,*

$$S_\lambda H = \mathbb{E}^{x_0} \int_0^{T_K} e^{-\lambda t} H(X_t) dt.$$

**Proof.** It is enough to consider  $H$  that are  $C^\infty$  with support in  $D - K$ . Let us kill the process on hitting  $K$ . Since  $H$  is 0 there, we can let the integrals run from 0 to  $\infty$ . Let  $u$  be defined by (4.6). Under a regular conditional probability for  $\mathcal{F}_t$ , the law of the process  $X_s \circ \theta_t$  is a solution to (2.10) started at  $X_t$ . Therefore by Proposition 4.7

$$\mathbb{E}_{\mathbb{P}} \left[ \int_0^\infty H(X_s \circ \theta_t) ds \mid \mathcal{F}_t \right] = \mathbb{E}^{X_t} \int_0^\infty H(X_s) ds = u(X_t).$$

We then have

$$\begin{aligned} (4.7) \quad S_\lambda u &= \mathbb{E}_{\mathbb{P}} \int_0^\infty e^{-\lambda t} u(X_t) dt \\ &= \mathbb{E}_{\mathbb{P}} \int_0^\infty e^{-\lambda t} \mathbb{E}_{\mathbb{P}} \left[ \int_0^\infty H(X_{s+t}) ds \mid \mathcal{F}_t \right] dt \\ &= \mathbb{E}_{\mathbb{P}} \int_0^\infty e^{-\lambda t} \int_t^\infty H(X_s) ds dt \\ &= \mathbb{E}_{\mathbb{P}} \int_0^\infty H(X_s) \int_0^s e^{-\lambda t} dt ds \\ &= \mathbb{E}_{\mathbb{P}} \int_0^\infty H(X_s) \frac{1 - e^{-\lambda s}}{\lambda} ds \\ &= \frac{1}{\lambda} u(x_0) - \frac{1}{\lambda} S_\lambda H, \end{aligned}$$

or  $S_\lambda H = u(x_0) - \lambda S_\lambda u$ . Define the operator  $R_\lambda$  by

$$(4.8) \quad R_\lambda f(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} f(X_t) dt.$$

We thus have  $u = R_0 H$  and so

$$(4.9) \quad S_\lambda H = R_0 H(x_0) - \lambda S_\lambda R_0 H.$$

Let

$$\Theta = \sup_{\|H\|_\infty \leq 1} |S_\lambda H - R_\lambda H(x_0)|.$$

Note  $\|R_\lambda H\|_\infty \leq \lambda^{-1}\|H\|_\infty$  and  $\|R_0 H\|_\infty \leq c_1\|H\|_\infty$ , where  $c_1 = \sup_y \mathbb{E}^y T_K$ . From the semigroup property of  $\mathbb{Q}^x$  (cf. [B], p. 19),

$$(4.10) \quad R_\lambda H(x_0) = R_0 H(x_0) - \lambda R_\lambda R_0 H(x_0).$$

Subtracting (4.10) from (4.9),

$$|S_\lambda H - R_\lambda H(x_0)| = |\lambda(S_\lambda R_0 H - R_\lambda R_0 H(x_0))| \leq \lambda \Theta \|R_0 H\|_\infty \leq \lambda \Theta c_1 \|H\|_\infty.$$

Taking the supremum over  $H$  with  $\|H\|_\infty \leq 1$ , if  $\lambda \leq 1/2c_1$ ,

$$\Theta \leq \lambda \Theta c_1 \leq \Theta/2.$$

Since

$$\Theta \leq \sup_{\|H\|_\infty \leq 1} (|S_\lambda H| + \|R_\lambda H\|_\infty) \leq 2/\lambda < \infty,$$

we have  $\Theta = 0$  or  $S_\lambda H = R_\lambda H(x_0)$ . □

**Proof of Theorem 2.2.** First suppose that  $d \geq 3$  and  $D$  satisfies (2.2). Recall the notation of Proposition 4.8 and that  $S_\lambda H = R_\lambda H(x_0)$ . By the uniqueness of the Laplace transform and the continuity of  $H(X_t)$  when  $H$  is continuous,  $\mathbb{E}_\mathbb{P} H(X_{t \wedge T_K}) = \mathbb{E}^{x_0} H(X_{t \wedge T_K})$ . As  $\rho \rightarrow 0$ , then  $T_K \rightarrow T_{\{x_0\}}$ . Since  $X_t$  behaves like a Brownian motion up until time  $T(\partial D)$ , then  $T_{\{x_0\}}$  is identically infinite. So  $\mathbb{E}_\mathbb{P} H(X_t) = \mathbb{E}^{x_0} H(X_t)$ . By standard arguments (see [SV2], Chapter 6), this implies that the finite dimensional distributions of  $X_t$  under  $\mathbb{P}$  and under  $\mathbb{Q}^{x_0}$  agree. Therefore  $\mathbb{P} = \mathbb{Q}^{x_0}$ .

Now let  $D$  be an arbitrary Lipschitz domain. By standard piecing-together arguments (see [SV2]) and (2.8), it suffices to show that for each  $x_0 \in \overline{D}$ , any two solutions  $\mathbb{P}_1$  and  $\mathbb{P}_2$  agree in a neighborhood of  $x_0$ . That is, if  $x_0 \in \overline{D}$ , there exists  $r > 0$  such that  $\mathbb{P}_1$  and  $\mathbb{P}_2$  agree on  $\mathcal{F}_{T(\partial B(x_0, r))}$ . Inside  $D$ ,  $X_t$  under both  $\mathbb{P}_1$  and  $\mathbb{P}_2$  behaves like ordinary Brownian motion, so we need only consider  $x_0 \in \partial D$ . Let a coordinate system and a domain  $D'$  satisfying (2.2) be chosen so that  $D'$  agrees with  $D$  in a neighborhood  $B(x_0, r) \cap D$  of  $x_0$ . Define  $\mathbb{P}'_i$  for  $i = 1, 2$  by

$$\mathbb{P}'_i(B \circ \theta_{T(\partial B(x_0, r))} \cap A) = \mathbb{E}_{\mathbb{P}'_i}(\mathbb{Q}_{D'}^{X(T(\partial B(x_0, r)))}(B); A), \quad A \in \mathcal{F}_{T(\partial B(x_0, r))}, \quad B \in \mathcal{F}_\infty,$$

where here  $(\mathbb{Q}_{D'}^x, X_t)$  is the law of reflecting Brownian motion in  $D'$  started at  $x$ . As in the discussion preceding Proposition 4.1,  $\mathbb{P}'_i$  is a solution to (2.10) in  $D'$  for  $i = 1, 2$ . By the uniqueness result for domains satisfying (2.2),  $\mathbb{P}'_1 = \mathbb{P}'_2 = \mathbb{Q}_{D'}^{x_0}$ . So if  $A \in \mathcal{F}_{T(\partial B(x_0, r))}$ , then  $\mathbb{P}_1(A) = \mathbb{P}'_1(A) = \mathbb{P}'_2(A) = \mathbb{P}_2(A)$ .

Finally we consider the case of  $d = 2$ . Let  $W_t$  be a one-dimensional Brownian motion reflecting at  $-1$  and  $1$  and independent of  $X_t$ . Then the law of  $(X_t, W_t)$  is a solution to (2.10) for the Lipschitz domain  $D \times (-1, 1)$ , and so is unique. The uniqueness of the law of  $X_t$  follows easily.  $\square$

## 5. Strong solutions.

A strong solution to (2.9) exists if there exists a process  $X_t$  satisfying (2.9) such that  $X$  is measurable with respect to the  $\sigma$ -fields of  $W$ . An interesting open problem is the following.

**Problem 5.1.** *Does there exist a strong solution to (2.9)?*

The reason for our interest is that if a strong solution exists, then in fact pathwise uniqueness holds for (2.9). That is, any two solutions to (2.9) must be identical. The proof of this is simple; cf. [K], Lemma 2.1.

**Proposition 5.2.** *Suppose a strong solution to (2.9) exists satisfying (2.10)(a)-(c). Then any two solutions to (2.9) that satisfy (2.10)(a)-(c) agree pathwise, a.s.*

**Proof.** Suppose  $dY_t = dW_t + (1/2)\nu(Y_t) dL_t$  and  $Y_t$  is a strong solution. Then there exists a measurable map  $F$  from  $C[0, \infty)$  to  $C[0, \infty)$  such that  $Y = F(W)$ . Let  $X_t$  be another solution. We have

$$(5.1) \quad W_t = Y_t - \frac{1}{2} \int_0^t \nu(Y_s) dL_s, \quad W_t = X_t - \frac{1}{2} \int_0^t \nu(X_s) dL_s.$$

The uniqueness in law (Theorem 2.2) says that the law of  $Y$  is equal to the law of  $X$ , so using (5.1) the law of the pair  $(Y, W)$  is equal to the law of the pair  $(X, W)$ . Since  $Y = F(W)$ , then  $X = F(W)$ , a.s., and we then conclude that  $X = F(W) = Y$ , a.s.  $\square$

**Remark.** We do not know the answer to Problem 5.1 even when  $D$  is a  $C^{1+\alpha}$  domain and even when the dimension  $d$  is 2. (The obvious conformal mapping argument does not appear to help). A  $C^{1+\alpha}$  domain is defined analogously to a Lipschitz domain, where we replace Lipschitz functions in the definition by functions whose gradient is Hölder continuous of order  $\alpha$ .

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