Moderate deviations for the range of planar random walks

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Abstract

Given a symmetric random walk in $\mathbb{Z}^2$ with finite second moments, let R_n be the range of the random walk up to time n. We study moderate deviations for $R_n - \mathbb{E}R_n$ and $\mathbb{E}R_n - R_n$. We also derive the corresponding laws of the iterated logarithm.

1 Introduction

Let X_i be symmetric i.i.d. random vectors taking values in $\mathbb{Z}^2$ with mean 0 and finite covariance matrix Γ, set $S_n = \sum_{i=1}^n X_i$, and suppose that no proper subgroup of $\mathbb{Z}^2$ supports the random walk S_n. For any random variable Y we will use the notation

$$\overline{Y} = Y - \mathbb{E}Y.$$

Let

$$R_n = \#\{S_1, \dots, S_n\} \tag{1.1}$$

be the range of the random walk up to time n. The purpose of this paper is to obtain moderate deviation results for $\overline{R}_n$ and $-\overline{R}_n$. With two exceptions,

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throughout this paper we only assume that the random walks have second moments. The two exceptions are Proposition 5.2 and Corollary 1.3, which supposes slightly more than two moments.

For moderate deviations of $\overline{R}_n$ we have the following. Let

$$\mathcal{H}(n) = \sum_{k=0}^{n} \mathbb{P}^0(S_k = 0). \tag{1.2}$$

Since the X_i have two moments, by (4.23) below,

$$\mathcal{H}(n) = \sum_{k=0}^{n} \mathbb{P}^0(S_k = 0) \sim \frac{\log n}{2\pi\sqrt{\det \Gamma}}$$

and

$$\mathcal{H}(n) - \mathcal{H}([n/b_n]) = \sum_{k=[n/b_n]+1}^{n} \mathbb{P}^0(S_k = 0) \sim \frac{\log b_n}{2\pi\sqrt{\det \Gamma}}.$$

Our first main result is the following.

Theorem 1.1 *Let $\{b_n\}$ be a positive sequence satisfying $b_n \to \infty$ and $\log b_n = o((\log n)^{1/2})$ as $n \to \infty$. There are two constants $C_1, C_2 > 0$ independent of the choice of the sequence $\{b_n\}$ such that*

$$\begin{aligned} -C_1 &\le \liminf_{n\to\infty} b_n^{-1} \log \mathbb{P}\Big\{\overline{R}_n \ge \frac{n}{\mathcal{H}(n)^2}(\mathcal{H}(n) - \mathcal{H}([n/b_n]))\Big\} \\ &\le \limsup_{n\to\infty} b_n^{-1} \log \mathbb{P}\Big\{\overline{R}_n \ge \frac{n}{\mathcal{H}(n)^2}(\mathcal{H}(n) - \mathcal{H}([n/b_n])\Big\} \le -C_2. \end{aligned} \tag{1.3}$$

Remark 1.2 The proof will show that C_2 in the statement of Theorem 1.1 is equal to the constant L given in Theorem 1.3 in [2]. We believe that C_1 is also equal to L, but we do not have a proof of this fact.

A more precise statement than Theorem 1.1 is possible when the X_i have slightly more than two moments.

Corollary 1.3 *Suppose $\mathbb{E}[\,|X_i|^2(\log^+(|X_i|))^{\frac{1}{2}+\delta}\,] < \infty$ for some $\delta > 0$. Let $\{b_n\}$ be a positive sequence satisfying $b_n \to \infty$ and $\log b_n = o((\log n)^{1/2})$ as*

$n \to \infty$. *There are two constants* $C_1, C_2 > 0$ *independent of the choice of the sequence* $\{b_n\}$ *such that*

$$-C_1 \le \liminf_{n\to\infty} b_n^{-\theta} \log \mathbb{P}\Big\{\overline{R}_n \ge 2\theta\pi\sqrt{\det\Gamma}\frac{n}{(\log n)^2}\log b_n\Big\}$$

$$\le \limsup_{n\to\infty} b_n^{-\theta} \log \mathbb{P}\Big\{\overline{R}_n \ge 2\theta\pi\sqrt{\det\Gamma}\frac{n}{(\log n)^2}\log b_n\Big\} \le -C_2 \tag{1.4}$$

for any $\theta > 0$.

Remark 1.4 The constants C_1, C_2 are the same as in the statement of Theorem 1.1. See Remark 1.2.

For b_n tending to infinity faster than the rate given in Theorem 1.1, e.g., $\log b_n = (\log n)^2$, then we are in the realm of large deviations. For Section 2 for some references to results on large deviations of the range.

For the moderate deviations of $-\overline{R}_n = \mathbb{E}R_n - R_n$ we have the following. Let $\kappa(2,2)$ be the smallest A such that

$$\|f\|_4 \le A\|\nabla f\|_2^{1/2}\|f\|_2^{1/2} \tag{1.5}$$

for all $f \in C^1$ with compact support. (This constant appeared in [2].)

Theorem 1.5 *Suppose* $b_n \to \infty$ *and* $b_n = o((\log n)^{1/5})$ *as* $n \to \infty$. *For* $\lambda > 0$

$$\lim_{n\to\infty}\frac{1}{b_n}\log \mathbb{P}\Big(-\overline{R}_n > \lambda\frac{nb_n}{(\log n)^2}\Big) = -(2\pi)^{-2}(\det\Gamma)^{-1/2}\kappa(2,2)^4\lambda.$$

Comparing Theorems 1.1 and 1.5, we see that the upper and lower tails of $\overline{R}_n$ are quite different. This is similar to the behavior of the distribution of the self-intersection local time of planar Brownian motion. This is not surprising, since LeGall, [24, Theorem 6.1], shows that $\overline{R}_n$, properly normalized, converges in distribution to the self-intersection local time; see also (2.2).

The moderate deviations of $\overline{R}_n$ are quite similar in nature to those of $-\overline{L}_n$, where L_n is the number of self-intersections of the random walk S_n; see [4]. Again, [24, Theorem 6.1] gives a partial explanation of this. However the case of the range is much more difficult than the corresponding results for intersection local times. The latter case can be represented as a quadratic functional of the path, which is amenable to the techniques of large deviation theory, while the range cannot be so represented. This has necessitated the development of several new tools, see in particular Sections 8 and 9, which we expect will have further applications in the study of the range of random walks.

Theorem 1.1 gives rise to the following LIL for $\overline{R}_n$.

Theorem 1.6

$$\limsup_{n\to\infty} \frac{\overline{R}_n}{n \log\log\log n/(\log n)^2} = 2\pi\sqrt{\det\Gamma}, \qquad \text{a.s.} \tag{1.6}$$

This result is an improvement of that in [5]; there it was required that the X_i be bounded random variables and the constant was not identified. Theorem 1.1 is a more precise estimate than is needed for Theorem 1.6; this is why Theorem 1.1 needs to be stated in terms of $\mathcal{H}(n)$ while Theorem 1.6 does not.

For an LIL for $-\overline{R}_n$ we have a different rate.

Theorem 1.7 *We have*

$$\limsup_{n\to\infty} \frac{-\overline{R}_n}{n \log\log n/(\log n)^2} = (2\pi)^{-2}\sqrt{\det\Gamma}\ \kappa(2,2)^4, \qquad \text{a.s.}$$

The study of the range of a lattice-valued (or $\mathbb{Z}^d$-valued) random walk has a long history in probability and the results show a strong dependence on the dimension d. See Section 2 for a brief history of the literature. The two dimensional case seems to be the most difficult; in one dimension no renormalization is needed (see [9]), while for $d \geq 3$ the tails are sub-Gaussian and have asymptotically symmetric behavior. In two dimensions, renormalization is needed and the tails have non-symmetric behavior. In this case, the central limit theorem was proved in 1986 in [24], while the first law of the iterated logarithm was not proved until a few years ago in [5].

We use results from the paper by Chen [8] in several places. This paper studies moderate deviations and laws of the iterated logarithm for the joint range of several independent random walks, that is, the cardinality of the set of points which are each visited by each of the random walks; see (3.2). Also related is the paper by Bass and Rosen [6], which is an almost sure invariance principle for the range. Our results in Theorems 1.1 and 1.5 are more precise than what can be obtained using [6]. We did not see how to derive our laws of the iterated logarithm from that paper; moreover in that paper $2+\delta$ moments were required.

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2 History

Perhaps the first result on the range of random walks is that of Dvoretzky and Erdös [13]. They proved a strong law of large numbers for the range of a random walk for a number of cases, including simple random walk in dimensions 2 and larger:

$$\frac{R_n}{\mathbb{E}R_n} \to 1, \qquad \text{a.s.} \tag{2.1}$$

This strong law was extended by Jain and Pruitt [20] to more general random walks, in particular, for any recurrent random walk in 2 dimensions. (The paper [20] improves upon the results in [19].) In each of these papers, the key is obtaining a good estimate on $\operatorname{Var} R_n$. Turning to central limit theorems, the paper [20] showed that in one dimension for finite variance walks, $R_n/\mathbb{E}R_n$ converges in law to the size of the range of one-dimensional Brownian motion if $\mathbb{E}X_1 = 0$ and

$$\frac{R_n - \mathbb{E}R_n}{(\operatorname{Var} R_n)^{1/2}}$$

converges in law to a standard normal if $\mathbb{E}X_1 \neq 0$. Jain and Orey [18] showed the same convergence for strongly transient random walks; in the case of finite variance with mean 0, this means that the dimension is 5 or larger. Jain and Pruitt [21] later established the analogous results for dimensions 3 and 4. In 3 dimensions, the variance of R_n is $O(n \log n)$, while in 4 dimensions the variance is $O(n)$. In both cases, the expectation of the range is $O(n)$. For random walks without moment conditions and for a weak invariance principle, see [23]. The central limit theorem for the range in two dimensions was not accomplished until LeGall [24]. In this paper LeGall proved the remarkable result that

$$\frac{R_n - \mathbb{E}R_n}{n/(\log n)^2} \to -4\pi^2\gamma_1, \tag{2.2}$$

where the convergence is in law, X_1 has the identity as its covariance matrix, and γ_1 is the renormalized self-intersection local time of a planar Brownian motion. (We will discuss intersection local times in a bit.) The result of LeGall can be extended to the case where X_1 has an arbitrary nondegenerate covariance matrix in a routine fashion.

The central limit theorem for random walks in the domain of attraction of a stable law was analyzed in [27]. It is noteworthy that in this case the results depend on the relationship between the dimension and the index of the stable law.

The law of the iterated logarithm for the range of random walks in 4 or more was established by Jain and Pruitt [22], where they showed that

$$\limsup_{n\to\infty} \frac{R_n - \mathbb{E}R_n}{(2n\log\log n)^{1/2}} = 1, \qquad \text{a.s.}$$

and that the corresponding $\liminf$ is -1 a.s.

For 2 and 3 dimensions, the law of the iterated logarithm was proved by Bass and Kumagai [5]. In 3 dimensions

$$\limsup_{n\to\infty} \frac{R_n - \mathbb{E}R_n}{(n\log n\log\log n)^{1/2}} = (p^2/\pi)\sqrt{\det\Gamma}, \qquad \text{a.s.}, \tag{2.3}$$

where Γ is the covariance matrix of X_1 and $p = \mathbb{P}(S_k \neq 0 \text{ for all } k)$. The corresponding $\liminf$ is the negative of this constant. These (and other laws of the iterated logarithm) are consequences of an invariance principle. In [5] it was proved that if the random walk is three dimensional, then there exists a one-dimensional Brownian motion such that

$$\frac{R_n - \mathbb{E}R_n}{(\sqrt{2}p^2/2\pi\sqrt{\det\Gamma})} - B_{n\log n} = O(\sqrt{n}(\log n)^{15/32}). \tag{2.4}$$

The law of the iterated logarithm for the range in 2 dimensions was a bit surprising:

$$\limsup_{n\to\infty} \frac{R_n - \mathbb{E}R_n}{n\log\log\log n/(\log n)^2} = c_1, \qquad \text{a.s.} \tag{2.5}$$

provided the random walk had bounded range and where c_1 is an unidentified constant. Among the results in the current paper is that we identify the constant, we remove the restriction of bounded range, and we determine the corresponding lim inf.

An almost sure invariance principle for $R_n - \mathbb{E}R_n$ in 4 or more dimensions was proved by Hamana [15]. There the result is that there exists a one-dimensional Brownian motion B_t such that for each $\lambda > 0$

$$\frac{R_n - \mathbb{E}R_n}{(\operatorname{Var} R_n)^{1/2}} - B_n = O(n^{2/5+\lambda}). \tag{2.6}$$

The almost sure invariance principle for random walks in 2 dimensions is more complicated. Bass and Rosen [6] showed that if the random walk has mean 0, has the identity as the covariance matrix, and has $2+\delta$ moments for some $\delta > 0$, then for each k

$$(\log n)^k \Big[\frac{R_n}{n} + \sum_{j=1}^{k}(-1)^j\Big(\frac{\log n}{2\pi} + c_X\Big)^{-j}\gamma_{j,n}\Big] \to 0, \qquad \text{a.s.} \tag{2.7}$$

where W_t is a 2-dimensional Brownian motion, c_X is a constant depending on the random walk, and $\gamma_{j,n}$ is the renormalized self-intersection local time of order j at time 1 of the Brownian motion $W_t^{(n)} = W_{nt}/\sqrt{n}$. The intuition behind this formula is the following: at time n, if the process hits a point it has already hit before, then $n - R_n$ increases by 1, and so does the number of self-intersections of the random walk. If the point that is hit again has only been hit once before, then the double self-intersections of the random walk increases by 1, but if this point has been hit a number of times, then the double self-intersections will go up more than 1, and this has to be compensated by subtracting off the number of triple self-intersections, and so on.

Large deviations for the range have been considered by Donsker and Varadhan [12], by Hamana [16], and by Hamana and Kesten [17]. These are related to estimates of the type

$$\mathbb{P}(R_n \geq \varphi(n)) \qquad \text{or} \qquad \mathbb{P}(R_n \leq \varphi(n))$$

for functions $\varphi(n)$ that grow relative quickly. By contrast, in our paper we look at

$$\mathbb{P}(R_n - \mathbb{E}R_n \geq \varphi(n))$$

for functions $\varphi(n)$ that grow not quite so quickly. The paper [12] determines

$$\lim_{n\to\infty} n^{d/(d+\alpha)} \log \mathbb{E} e^{-\lambda R_n},$$

when X_1 is in the domain of attraction of a symmetric stable process of index α. This can be used to obtain information on $\mathbb{P}(R_n \leq \varphi(n))$. It is shown in [17] that

$$\lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}(R_n \geq nx)$$

exists, while [16] examines

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E} e^{\lambda R_n}.$$

Related to our paper is one by Chen [8], which looks at moderate deviations of the joint range J_n, the cardinality of the set of points that are each visited by each of p independent random walks.

We will be much briefer concerning the literature of intersection local times. There are a large number of papers, but we only mention those relevant to this paper. Given two independent Brownian motions V_t, W_t in two dimensions, formally the intersection local time is the quantity

$$\int_0^t \int_0^t \delta_0(V_s - W_u)\, du\, ds,$$

which is a measure of how often the two Brownian motions intersect each other; here δ_0 is the delta function. To give a meaning to this, we define the intersection local time by

$$\lim_{\varepsilon\to 0} \int_0^t \int_0^t p_\varepsilon(V_s - W_u)\, ds\, du, \tag{2.8}$$

where p_ε is a suitable approximation to the identity, for example, the density of a two-dimensional Brownian motion at time ε. If one now wants to define self-intersection local time of one planar Brownian motion, one cannot simply replace V_s by W_s. If one does, one gets an identically infinite random process. Varadhan [30] proved that provided one renormalizes properly, one can get a finite limit. There are a number of renormalizations possible. We use the following: let

$$\gamma_\varepsilon(t) = \int_0^t \int_0^u p_\varepsilon(W_s - W_u)\, ds\, du, \tag{2.9}$$

and let

$$\gamma_t = \lim_{\varepsilon \to 0}[\gamma_\varepsilon(t) - \mathbb{E}\gamma_\varepsilon(t)]. \tag{2.10}$$

The limit exists almost surely, and is called renormalized self-intersection local time for planar Brownian motion.

LeGall [26] proved that

$$\mathbb{E}e^{\lambda\gamma_1} \tag{2.11}$$

is finite for all negative λ and for all sufficiently small positive λ and is infinite for all sufficiently large positive λ. The critical value λ_0 at which the expectation switches from being finite to infinite was found by Bass and Chen [2]. It turns out that the expectation in (2.11) is finite if $\lambda < \kappa(2,2)^{-4}$ and infinite if $\lambda > \kappa(2,2)^{-4}$, where $\kappa(2,2)$ is the best constant in a Gagliardo-Nirenberg inequality; see (1.5).

We mention that our paper [4] has moderate deviations results for the number of self-intersections of random walks which are similar in form to those of this paper, although the proofs are different.

3 Overview

Before we outline our methods of proof, let us mention two techniques that are useful in the study of the range. The first is rewriting the range as a sum, the second is writing the range of a single random walk in terms of the joint range of two random walks.

To explain the first technique, let $T_y = \min\{k : S_k = y\}$, the first time the random walk hits y. The key observation is that

$$R_n = \sum_{y \in \mathbb{Z}^2} 1_{(T_y \le n)}. \tag{3.1}$$

This equality is simply the fact that the sum on the right is the number of points that have been hit by time n, which is the same as the range.

From (3.1) one can estimate moments. For example,

$$\mathbb{E}R_n = \sum_{x \in \mathbb{Z}^2} \mathbb{P}(T_x \le n),$$

and therefore a good estimate of the mean follows from a good estimate on $\mathbb{P}(T_y \le n)$. One can also obtain higher moments in this way. Thus,

$$\begin{aligned}\mathbb{E}R_n^2 &= \mathbb{E}\left[\sum_{x\in\mathbb{Z}^2}\sum_{y\in\mathbb{Z}^2} 1_{(T_x\le n)}1_{(T_y\le n)}\right]\\ &= \sum_{x\in\mathbb{Z}^2}\sum_{y\in\mathbb{Z}^2}\mathbb{P}(T_x\le n, T_y\le n).\end{aligned}$$

Let S, S' be two independent random walks. The joint range J_n is defined by

$$J_n = \#\left(\{S_1,\dots,S_n\}\cap\{S'_1,\dots,S'_n\}\right). \tag{3.2}$$

The reason J_n is more manageable than R_n is that if both random walks start at the same point, then

$$\mathbb{E}J_n = O\Big(\frac{n}{(\log n)^2}\Big), \qquad \operatorname{Var} J_n = O\Big(\frac{n^2}{(\log n)^4}\Big), \tag{3.3}$$

while

$$\mathbb{E}R_n = O\Big(\frac{n}{\log n}\Big), \qquad \operatorname{Var} R_n = O\Big(\frac{n^2}{(\log n)^4}\Big). \tag{3.4}$$

We thus see that the expectation of the range is large compared to the standard deviation of the range, while the expectation of the joint range is comparable to the standard deviation of the joint range. This shows that the joint range is somewhat less singular than the range.

To exploit this, if n is even we can break $\{S_1,\dots,S_n\}$ into the two sets $\{S_1,\dots,S_{n/2}\}$ and $\{S_{(n/2)+1},\dots,S_n\}$, and therefore

$$\begin{aligned}R_n = \#\{S_1,\dots,S_{n/2}\} &+ \#\{S_{(n/2)+1},\dots,S_n\}\\ &- \#(\{S_1,\dots,S_{n/2}\}\cap\{S_{(n/2)+1},\dots,S_n\}).\end{aligned} \tag{3.5}$$

The first two terms are each equal in law to $R_{n/2}$ and the third term is equal in law to $J_{n/2}$, where, however, the two independent pieces might not both start at the same point.

We can repeat by decomposing each of the first two terms on the right of (3.5) into two pieces, and continue to get a decomposition of the range into the sum of joint ranges plus some small leftover terms. We can also do other decompositions by letting $I_1, \dots, I_m$ be disjoint subintervals of $N = \{1, \dots, n\}$ whose union is N, letting $S(I_i) = \{S_k : k \in I_i\}$, and writing the range R_n

$$R_n = \sum_i \# S(I_i) - \sum_{i<j} \# (S(I_i) \cap S(I_j)). \tag{3.6}$$

We use a number of such decompositions in this paper.

Let us now turn to overviews of some of the proofs in this paper. We first look at moderate deviations of $R_n - \mathbb{E}R_n$, and examine the upper bound. By LeGall's central limit theorem (see (2.2)),

$$A_n = \frac{R_n - \mathbb{E}R_n}{n/(\log n)^2}$$

converges in law to a multiple of γ_1, the self-intersection local time of planar Brownian motion. Therefore if we have suitable exponential bounds on A_n, then by dominated convergence

$$\mathbb{E}e^{\lambda A_n} \to \mathbb{E}e^{c\lambda\gamma_1}. \tag{3.7}$$

One has to be careful in that the right side is finite for some λ but not all λ. By using a decomposition such as the one described in (3.6), this turns out to be all we need to get the upper bound on the moderate deviations. The key to the argument is thus getting appropriate exponential bounds on A_n. By a Taylor series expansion of $e^{\lambda x}$, it is enough to get bounds on the moments of A_n. This is done by means of the decompositions in terms of joint ranges described in (3.6).

Let us now look at the lower bound. We break $N = \{1, \dots, n\}$ into subintervals I_i of length approximately b_n. Using the Markov property, we can show that the path of $\{S_k : 1 \le k \le n\}$ is reasonably close to a straight line with a certain probability. The lower bound comes from such paths. We use the decomposition (3.6). For the paths that we are considering, $S(I_i) \cap S(I_j) = \emptyset$ if $|i - j| > 1$. So in (3.6) we have the sum $\sum_i \# S(I_i)$, which is (almost) the sum of i.i.d. random variables and the sum $\sum_i \# (S(I_i) \cap S(I_{i+1}))$, which can be broken up into the two sums of (almost) i.i.d. random variables. Standard techniques for the sums of independent random variables are then applied. We remark that the quantity

$$\mathbb{E}R_n - \sum_i \mathbb{E}[\#S(I_i)]$$

plays an important role in both the upper and lower bound estimates.

Next we give an overview for the moderate deviations of $\mathbb{E}R_n - R_n$. The upper bound is handled by using a decomposition of the form (3.5). We show that the dominant contribution comes from the intersections of independent ranges, which has been worked out in [8, Theorem 1]. The lower bound is more subtle and requires a new approach. We use a smoothed version of the range, obtained by convolving the indicator function in (3.1) with a smooth function of compact support. We exploit the regularity of the smoothed range to obtain its moderate deviations, and then develop estimates to show that the moderate deviations for the true range can be approximated by those of the smoothed range.

Lastly, we look at the laws of the iterated logarithms. Here we use a combination of two ideas. Theorems 1.1 and 1.5 allow us to obtain the laws of the iterated logarithm along exponential sequences. What remains is to fill in the gaps, that is, we need estimates on

$$\mathbb{P}(\max_{i\le n}(R_i - \mathbb{E}R_i) > \lambda n \log\log\log n/(\log n)^2)$$

and the analogous probabilities for $\max_{i\le n}(\mathbb{E}R_i - R_i)$. We obtain these by using the technique of metric entropy; this is also known as chaining, and is based on the method used in the proof of Kolmogorov's continuity criterion (cf. [1, I.3.11]).

4 Preliminaries

As usual we set $\mathcal{F}_n = \sigma(S_1, \dots, S_n)$. For $x \in \mathbb{Z}^d$, let $f_n^x(S)$ be a functional of the random walk S which is non-negative and adapted, i.e $f_n^x(S) \in \mathcal{F}_n$. Let θ_n be the usual shift operators of Markov process theory, so that $S_k \circ \theta_n = S_{k+n}$. We will say that $f_n^x(S)$ is a subadditive functional if for any k, n

$$f_{n+k}^x(S) \le f_n^x(S) + f_k^x(S) \circ \theta_n. \tag{4.1}$$

We will say that $f_n^x(S)$ is spatially homogeneous if for any n and $x, z \in \mathbb{Z}^d$

$$f_n^{x-z}(S) = f_n^x(S+z). \tag{4.2}$$

$f_n^x(S) = 1_{\{x \in S[1,n]\}}$ is an example of a spatially homogeneous subadditive functional. Note that the range $R_n = \sum_{x \in \mathbb{Z}^d} 1_{\{x \in S[1,n]\}}$.

The following Theorem 4.1 is a generalization of Theorem 6 and Corollary 1 of [8].

Theorem 4.1 *Let $f_n^x(S)$ be a spatially homogeneous subadditive functional of S and for p independent copies $S^{(1)}, \ldots, S^{(p)}$ of S set*

$$F_n = \sum_{x \in \mathbb{Z}^d} \prod_{j=1}^p f_n^x(S^{(j)}). \tag{4.3}$$

Then for any integers $n_1, \cdots, n_\ell \geq 1$,

$$\big(\mathbb{E}F^m_{n_1+\cdots+n_\ell}\big)^{1/p} \leq \sum_{\substack{k_1+\cdots+k_\ell=m \\ k_1,\cdots,k_\ell \geq 0}} \frac{m!}{k_1! \cdots k_\ell!} \big(\mathbb{E}F^{k_1}_{n_1}\big)^{1/p} \cdots \big(\mathbb{E}F^{k_\ell}_{n_\ell}\big)^{1/p}, \tag{4.4}$$

and for any $\theta > 0$,

$$\sum_{m=0}^\infty \frac{\theta^m}{m!} \big(\mathbb{E}F^m_{n_1+\cdots+n_\ell}\big)^{1/p} \leq \prod_{i=1}^\ell \sum_{m=0}^\infty \frac{\theta^m}{m!} \big(\mathbb{E}F^m_{n_i}\big)^{1/p}. \tag{4.5}$$

Proof. (4.5) follows immediately from (4.4), and by induction it suffices to prove (4.4) in the case $\ell = 2$. Using the standard notation

$$\|\psi(x_1, \ldots, x_m)\|_{L^p((\mathbb{Z}^d)^m)} = \bigg(\sum_{(x_1,\ldots,x_m) \in (\mathbb{Z}^d)^m} |\psi(x_1, \ldots, x_m)|^p \bigg)^{1/p}$$

we note that

$$\big(\mathbb{E}F^m_n\big)^{1/p} = \|\mathbb{E}\big(\prod_{i=1}^m f_n^{x_i}(S)\big)\|_{L^p((\mathbb{Z}^d)^m)}. \tag{4.6}$$

Then by subadditivity

$$
\begin{aligned}
(4.7)\qquad & \big(\mathbb{E}F^m_{n_1+n_2}\big)^{1/p} \\
&= \|\mathbb{E}\big(\prod_{i=1}^m f^{x_i}_{n_1+n_2}(S)\big)\|_{L^p((\mathbb{Z}^d)^m)} \\
&\le \sum_{A\subseteq[1,m]} \|\mathbb{E}\big(\prod_{i\in A} f^{x_i}_{n_1}(S)\prod_{j\in A^c} f^{x_j}_{n_2}(S)\circ\theta_{n_1}\big)\|_{L^p((\mathbb{Z}^d)^m)}.
\end{aligned}
$$

By the Markov property

$$
(4.8)\qquad \mathbb{E}\big(\prod_{i\in A} f^{x_i}_{n_1}(S)\prod_{j\in A^c} f^{x_j}_{n_2}(S)\circ\theta_{n_1}\big) = \mathbb{E}\big(\prod_{i\in A} f^{x_i}_{n_1}(S)\widetilde{\mathbb{E}}^{S_{n_1}}\big(\prod_{j\in A^c} f^{x_j}_{n_2}(\widetilde{S})\big)\big),
$$

where for ease of notation we have used $\widetilde{S},\widetilde{\mathbb{E}}$ to denote an independent copy of S and its expectation operator. Using (4.8) and spatially homogeneity we have

$$
\begin{aligned}
(4.9)\qquad & \|\mathbb{E}\big(\prod_{i\in A} f^{x_i}_{n_1}(S)\prod_{j\in A^c} f^{x_j}_{n_2}(S)\circ\theta_{n_1}\big)\|_{L^p((Z^d)^m)} \\
&= \|\mathbb{E}\big(\prod_{i\in A} f^{x_i}_{n_1}(S)\widetilde{\mathbb{E}}\big(\prod_{j\in A^c} f^{x_j-S_{n_1}}_{n_2}(\widetilde{S})\big)\big)\|_{L^p((Z^d)^m)} \\
&= \Big\{\mathbb{E}\Big(\sum_{x_i,i\in[1,m]}\prod_{k=1}^p\prod_{i\in A} f^{x_i}_{n_1}(S^{(k)})\widetilde{\mathbb{E}}\big(\prod_{j\in A^c} f^{x_j-S^{(k)}_{n_1}}_{n_2}(\widetilde{S})\big)\Big)\Big\}^{1/p} \\
&= \Big\{\mathbb{E}\Big(F^{|A|}_{n_1}\sum_{x_j,j\in A^c}\prod_{k=1}^p\widetilde{\mathbb{E}}\big(\prod_{j\in A^c} f^{x_j-S^{(k)}_{n_1}}_{n_2}(\widetilde{S})\big)\Big)\Big\}^{1/p}
\end{aligned}
$$

and by Hölder's inequality and then spatial homogeneity

$$
\begin{aligned}
(4.10)\qquad & \sum_{x_j,j\in A^c}\prod_{k=1}^p\widetilde{\mathbb{E}}\big(\prod_{j\in A^c} f^{x_j-S^{(k)}_{n_1}}_{n_2}(\widetilde{S})\big)\Big) \\
&\le \prod_{k=1}^p \|\widetilde{\mathbb{E}}\big(\prod_{j\in A^c} f^{x_j-S^{(k)}_{n_1}}_{n_2}(\widetilde{S})\|_{L^p((Z^d)^{|A^c|})} \\
&= \|\widetilde{\mathbb{E}}\big(\prod_{j\in A^c} f^{x_j}_{n_2}(\widetilde{S})\big)\big)\|^p_{L^p((Z^d)^{|A^c|})} = \big(\mathbb{E}F^{|A^c|}_{n_2}\big).
\end{aligned}
$$

Thus we have shown that

$$
\begin{aligned}
(4.11) \qquad & \|\mathbb{E}\big(\prod_{i\in A} f_{n_1}^{x_i}(S) \prod_{j\in A^c} f_{n_2}^{x_j}(S)\circ\theta_{n_1}\big)\|_{L^p((Z^d)^m)} \\
& \le \big(\mathbb{E}F_{n_1}^{|A|}\big)^{1/p}\big(\mathbb{E}F_{n_2}^{|A^c|}\big)^{1/p}.
\end{aligned}
$$

This proves (6.20) in the case $l = 2$ and hence our lemma now follows. □

Remark 4.2 There is a variation of the lemma which is useful. For a fixed integer $t > 0$, let $\mathcal{F}'_n = \mathcal{F}_{tn}$. Let now $f_n^x(S)$, be adapted to $\mathcal{F}'_n$, i.e. $f_n^x(S) \in \mathcal{F}'_n$ for all n, and additive with respect to $\mathcal{F}'_n$, i.e. for any k, n

$$
(4.12) \qquad f_{n+k}^x(S) = f_n^x(S) + f_k^x(S)\circ\theta_{tn}.
$$

We assume as before that $f_n^x(S)$ is spatially homogeneous. The same proof shows that Theorem 4.1 continues to hold. □

Remark 4.3 Let $\mathbb{E}^{y_1,\ldots,y_p}(\cdot)$ denote expectations with respect to the independent random walks $S^{(1)},\ldots,S^{(p)}$ started at $y_1,\ldots,y_p$ respectively. It follows by an argument similar to that of (4.11) that

$$
(4.13) \qquad \sup_{y_1,\ldots,y_p} \mathbb{E}^{y_1,\ldots,y_p}\left(F_n^m\right) = \mathbb{E}(F_n^m).
$$

In more detail, by Hölder's inequality and spatial homogeneity

$$
\begin{aligned}
(4.14) \qquad \sup_{y_1,\ldots,y_p} \mathbb{E}^{y_1,\ldots,y_p}\left(F_n^m\right) &= \|\prod_{j=1}^{p}\mathbb{E}\big(\prod_{i=1}^{m} f_n^{x_i}(S+y_j)\big)\|_{L^1((\mathbb{Z}^d)^m)} \\
&\le \prod_{j=1}^{p}\|\mathbb{E}\big(\prod_{i=1}^{m} f_n^{x_i}(S+y_j)\big)\|_{L^p((\mathbb{Z}^d)^m)} \\
&= \|\mathbb{E}\big(\prod_{i=1}^{m} f_n^{x_i}(S)\big)\|^p_{L^p((\mathbb{Z}^d)^m)} = \mathbb{E}(F_n^m).
\end{aligned}
$$

□

Set

$$\varphi(u) = E\left(e^{iu\cdot X_1}\right). \tag{4.15}$$

It follows from our assumptions that $\varphi(u) \in C^2$, $\frac{\partial}{\partial u_i}\varphi(0) = 0$ and $\frac{\partial^2}{\partial u_i \partial u_j}\varphi(0) = -E\left(X_1^{(i)} X_1^{(j)}\right)$ where $X_1 = (X_1^{(1)},\, X_1^{(2)})$, so that for some $\delta > 0$

$$\varphi(u) = 1 - E\left((u\cdot X_1)^2\right)/2 + o(|u|^2), \qquad |u| \leq \delta. \tag{4.16}$$

Then for some $c_1 > 0$ and $\delta > 0$ sufficiently small

$$|\varphi(u)| \leq e^{-c_1|u|^2}, \qquad |u| \leq \delta. \tag{4.17}$$

Strong aperiodicity implies that $|\varphi(u)| < 1$ for $u \neq 0$ and $u \in [-\pi,\pi]^2$. In particular, we can find $b_0 < 1$ such that $|\varphi(u)| \leq b_0$ for $\delta \leq |u|$ and $u \in [-\pi,\pi]^2$. But clearly we can choose $c_2 > 0$ so that $b_0 \leq e^{-c_2|u|^2}$ for $u \in [-\pi,\pi]^2$. Setting $c = \min(c_1, c_2) > 0$ we then have

$$|\varphi(u)| \leq e^{-c|u|^2}, \qquad u \in [-\pi,\pi]^2. \tag{4.18}$$

Then with $C = [-\pi,\pi]^2$

$$\begin{aligned} \lim_{n\to\infty} n\mathbb{P}(S_n = 0) &= \lim_{n\to\infty} \frac{n}{2\pi}\int_C (\varphi(u))^n\, du \\ &= \lim_{n\to\infty} \frac{1}{2\pi}\int_{\sqrt{n}C} (\varphi(u/\sqrt{n}))^n\, du. \end{aligned} \tag{4.19}$$

By (4.16) we see that $(\varphi(u/\sqrt{n}))^n \to \exp(-u\cdot\Gamma u/2)$ on $|u| \leq \delta\sqrt{n}$, and by (4.18) we can apply the dominated convergence theorem to obtain

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{|u|\leq\sqrt{n}\delta} (\varphi(u/\sqrt{n}))^n\, du = \frac{1}{2\pi}\int_{\mathbb{R}^2} \exp(-u\cdot\Gamma u/2)\, du = \frac{1}{2\pi\sqrt{\det\Gamma}}. \tag{4.20}$$

On the other hand, by the above, for some $b_0 < 1$

$$\limsup_{n\to\infty} \frac{1}{2\pi}\left|\int_{\sqrt{n}\delta<|u|\leq\sqrt{n}\pi} (\varphi(u/\sqrt{n}))^n\, du\right| \leq \frac{1}{2\pi}\int_{\sqrt{n}C} b_0^n\, du = 0. \tag{4.21}$$

Thus

$$\lim_{n\to\infty} n\mathbb{P}(S_n = 0) = \frac{1}{2\pi\sqrt{\det\Gamma}} \tag{4.22}$$

and it follows from this that

$$\mathcal{H}(n) = \sum_{k=0}^{n} \mathbb{P}^0(S_k = 0) \sim \frac{\log n}{2\pi\sqrt{\det\Gamma}} \tag{4.23}$$

and

$$\mathcal{H}(n) - \mathcal{H}(m) \sim \frac{\log(n/m)}{2\pi\sqrt{\det\Gamma}} \tag{4.24}$$

as n and m tend to infinity.

We write $S(I)$ for $\{S_k : k \in I\}$. Let $S^{(i)}$, $i = 1, \ldots, p$ be p independent copies of S. Let

$$J_n = \#\big\{S^{(1)}[1,n] \cap \cdots \cap S^{(p)}[1,n]\big\} \quad n = 1, 2, \cdots .$$

This can be written as

$$J_n = \sum_{y\in\mathbb{Z}^2} \prod_{k=1}^{p} 1_{\{y\in S^{(k)}[1,n]\}} \tag{4.25}$$

and therefore

$$\mathbb{E}(J_n) = \sum_{y\in\mathbb{Z}^2} \left(\mathbb{P}(y \in S[1,n])\right)^p . \tag{4.26}$$

$\mathbb{E}(J_n)$ can be bounded as follows. First write the obvious bound

$$\begin{aligned} &\sum_{y\in\mathbb{Z}^2} \left(\mathbb{E}\left(1_{\{y\in S[1,n]\}} \sum_{i=1}^{2n} 1_{\{S_i=y\}}\right)\right)^p \\ &\leq \sum_{y\in\mathbb{Z}^2} \left(\mathbb{E}\left(\sum_{i=1}^{2n} 1_{\{S_i=y\}}\right)\right)^p . \end{aligned} \tag{4.27}$$

Applying the Markov property at T_y, the first hitting time of y, we see that

$$\mathbb{E}\left(1_{\{y\in S[1,n]\}} \sum_{i=1}^{2n} 1_{\{S_i=y\}}\right) \;\geq\; \mathbb{P}(y \in S[1,n])\mathcal{H}(n). \tag{4.28}$$

Hence from (4.26)-(4.28)

$$\mathbb{E}(J_n)\mathcal{H}^p(n) \le \sum_{y\in\mathbb{Z}^2}\left(\mathbb{E}\left(\sum_{i=1}^{2n} 1_{\{S_i=y\}}\right)\right)^p \tag{4.29}$$

$$= \sum_{y\in\mathbb{Z}^2}\sum_{i_1,\dots,i_p=1}^{2n}\frac{1}{(2\pi)^p}\int_{(\mathbf{T}^2)^p} e^{i(\sum_{j=1}^p u_j)\cdot y}\prod_{j=1}^p(\varphi(u_j))^{i_j}\,du_j$$

$$= \sum_{i_1,\dots,i_p=1}^{2n}\frac{1}{(2\pi)^{p-1}}\int_{(\mathbf{T}^2)^{p-1}}(\varphi(-\sum_{j=1}^{p-1}u_j))^{i_p}\prod_{j=1}^{p-1}(\varphi(u_j))^{i_j}\,du_j$$

where the last step used Fourier inversion for u_p. By (4.18) we thus have

$$\mathbb{E}(J_n)\mathcal{H}^p(n) \le \sum_{i_1,\dots,i_p=1}^{2n}\frac{1}{(2\pi)^{p-1}}\int_{(\mathbb{R}^2)^{p-1}} e^{-ci_p|\sum_{j=1}^{p-1}u_j|^2}\prod_{j=1}^{p-1}e^{-ci_j|u_j|^2}\,du_j. \tag{4.30}$$

Let

$$F_j = e^{-ci_p|\sum_{m=1}^{p-1}u_m|^2}\prod_{l=1,l\neq j}^{p-1}e^{-ci_l|u_l|^2}, \qquad 1\le j\le p-1 \tag{4.31}$$

and

$$F_p = \prod_{l=1}^{p-1}e^{-ci_l|u_l|^2}. \tag{4.32}$$

Then we can write (4.30) as

$$\mathbb{E}(J_n)\mathcal{H}^p(n) \le \sum_{i_1,\dots,i_p=1}^{2n}\frac{1}{(2\pi)^{p-1}}\int_{(\mathbb{R}^2)^{p-1}}\prod_{j=1}^{p}F_j^{1/(p-1)}\prod_{l=1}^{p-1}du_l. \tag{4.33}$$

Then by the multiple Hölder inequality

$$\mathbb{E}(J_n)\mathcal{H}^p(n) \le C\sum_{i_1,\dots,i_p=1}^{2n}\prod_{j=1}^{p}\|F_j^{1/(p-1)}\|_p. \tag{4.34}$$

It is easy to obtain the bounds

$$\|F_j^{1/(p-1)}\|_p \le C i_p^{-1/p} \prod_{l=1, l\neq j}^{p-1} i_l^{-1/p}, \quad 1 \le j \le p-1 \tag{4.35}$$

$$\|F_p^{1/(p-1)}\|_p \le C \prod_{l=1}^{p-1} i_l^{-1/p} \tag{4.36}$$

so that

$$\mathbb{E}(J_n)\mathcal{H}^p(n) \le C \sum_{i_1,\dots,i_p=1}^{2n} \prod_{j=1}^{p} i_j^{-(p-1)/p} = C\left(\sum_{i=1}^{2n} i^{-(p-1)/p}\right)^p \le Cn. \tag{4.37}$$

Thus we have

$$\mathbb{E}(J_n) \le \frac{Cn}{(\log n)^p}, \quad n = 1, \cdots \tag{4.38}$$

for some $C < \infty$.

We next note using (4.25) that

$$\begin{aligned} \mathbb{E}(J_n^k) &= \sum_{y_1,\dots,y_k \in \mathbb{Z}^2} \left(\mathbb{E}\left(\prod_{i=1}^{k} 1_{\{y_i \in S[1,n]\}}\right)\right)^p \\ &= \sum_{y_1,\dots,y_k \in \mathbb{Z}^2} \left(\sum_{\pi} \mathbb{P}\left(T_{y_{\pi(1)}} \le T_{y_{\pi(2)}} \le \cdots \le T_{y_{\pi(k)}} \le n\right)\right)^p \end{aligned} \tag{4.39}$$

where the inner sum is over all permutations π of $\{1,\dots,k\}$. Then by Hölder's inequality

$$\begin{aligned} &\mathbb{E}(J_n^k) \\ &\le (k!)^{(p-1)} \sum_{y_1,\dots,y_k \in \mathbb{Z}^2} \sum_{\pi} \left(\mathbb{P}\left(T_{y_{\pi(1)}} \le T_{y_{\pi(2)}} \le \cdots \le T_{y_{\pi(k)}} \le n\right)\right)^p \\ &= (k!)^p \sum_{y_1,\dots,y_k \in \mathbb{Z}^2} \left(\mathbb{P}\left(T_{y_1} \le T_{y_2} \le \cdots \le T_{y_k} \le n\right)\right)^p. \end{aligned} \tag{4.40}$$

Applying the Markov property at time $T_{y_{k-1}}$ we obtain

$$(4.41) \sum_{y_1,\dots,y_k\in\mathbb{Z}^2} (\mathbb{P}(T_{y_1} \le T_{y_2} \le \cdots \le T_{y_k} \le n))^p$$
$$\le \sum_{y_1,\dots,y_{k-1}\in\mathbb{Z}^2} \left(\mathbb{P}\left(T_{y_1} \le T_{y_2} \le \cdots \le T_{y_{k-1}} \le n\right)\right)^p \sum_{y_k\in\mathbb{Z}^2} (\mathbb{P}^{y_{k-1}}(T_{y_k} \le n))^p$$
$$= \sum_{y_1,\dots,y_{k-1}\in\mathbb{Z}^2} \left(\mathbb{P}\left(T_{y_1} \le T_{y_2} \le \cdots \le T_{y_{k-1}} \le n\right)\right)^p \mathbb{E}(J_n).$$

By induction we get

$$\mathbb{E}(J_n^k) \le (k!)^p (\mathbb{E}J_n)^k, \qquad k = 0, 1, \cdots. \tag{4.42}$$

The following Lemma, which is Lemma 1 in [3], is used in the next section.

Lemma 4.4 *Let $0 < p \le 1$ and let $\{Y_k(\zeta)\}_{k\ge1}$ be a family (indexed by ζ) of sequences of i.i.d. real valued random functions such that $E(Y_k(\zeta)) = 0$ and*

$$\lim_{\theta\to0} \sup_{\zeta} Ee^{\theta|Y_1(\zeta)|^p} = 1. \tag{4.43}$$

Then for some $\lambda > 0$,

$$\sup_{n,\zeta} E \exp\left\{\lambda\Big|\sum_{k=1}^n Y_k(\zeta)/\sqrt{n}\Big|^p\right\} < \infty. \tag{4.44}$$

Proof. Let $\psi_p(x) = e^{x^p} - 1$ for large x and linear near the origin so that $\psi_p(x)$ is convex. We use $\|\cdot\|_{\psi_p}$ to denote the norm of the Orlicz space L_{ψ_p} with Young's function ψ_p. The assumption (4.43) of our Lemma implies that for some $M < \infty$

$$\sup_{\zeta} \|Y_1(\zeta)\|_{\psi_p} \le M. \tag{4.45}$$

By Theorem 6.21 of [28], if ξ_k are i.i.d. copies of a mean zero random variable $\xi_1 \in L_{\psi_p}$ then for some constant K_p depending only on p

$$\Big\|\sum_{k=1}^n \xi_k\Big\|_{\psi_p} \le K_p \left(\Big\|\sum_{k=1}^n \xi_k\Big\|_{L_1} + \Big\|\max_{1\le k\le n} |\xi_k|\Big\|_{\psi_p}\right).$$

Using Prop 4.3.1 of [14], for some constant C_p depending only on p

$$\left\| \max_{1\le k\le n} |\xi_k| \right\|_{\psi_p} \le C_p(\log n)\|\xi_1\|_{\psi_p}.$$

Since the ξ_k are i.i.d. and mean zero

$$\| \sum_{k=1}^n \xi_k \|_{L_1} \le \| \sum_{k=1}^n \xi_k \|_{L_2} \le \sqrt{n}\|\xi_1\|_{L_2}.$$

Thus we have

$$\left\| \sum_{k=1}^n \xi_k/\sqrt{n} \right\|_{\psi_p} \le D_p \left(\|\xi_1\|_{L_2} + \frac{\log n}{\sqrt{n}} \|\xi_1\|_{\psi_p} \right)$$

for some constant D_p depending only on p. Our Lemma follows immediately from this. □

5 Moments of the range

In this section we first give an estimate for the expectation of the range. The next result is contained in [27, Theorem 6.9].

Lemma 5.1

$$\mathbb{E}R_n = \frac{n}{\mathcal{H}(n)} + \frac{1}{2\pi\sqrt{\det\Gamma}}\frac{n}{\mathcal{H}(n)^2}(1+o(1)), \tag{5.1}$$

where $\mathcal{H}$ is defined in (1.2).

Proof: Let

$$u_n = \mathbb{P}(S_n = 0), \qquad r_n = \mathbb{P}(\mathcal{T}_0 > n) \tag{5.2}$$

where $\mathcal{T}_0 = \min\{n \ge 1 : S_n = 0\}$. Then

$$\mathbb{E}R_n = \sum_{i=0}^n r_i. \tag{5.3}$$

By considering the last visit to 0 before time n we see that

$$\sum_{i=0}^{n} u_i r_{n-i} = 1. \tag{5.4}$$

Since r_i is non-increasing this shows that

$$r_n \leq \left(\sum_{i=0}^{n} u_i \right)^{-1} = \frac{1}{\mathcal{H}(n)}. \tag{5.5}$$

We next obtain a lower bound for r_n. Fix $\epsilon > 0$. Then by (5.4)

$$\left(\sum_{i=0}^{[\epsilon n]} u_i \right) r_n \geq 1 - \sum_{i=[\epsilon n]+1}^{n+[\epsilon n]} u_i r_{n+[\epsilon n]-i} = 1 - \sum_{i=0}^{n-1} u_{n+[\epsilon n]-i} r_i, \tag{5.6}$$

and therefore, using also (5.5),

$$r_n \geq \frac{1}{\mathcal{H}([\epsilon n])} - \frac{1}{\mathcal{H}([\epsilon n])} \sum_{i=0}^{n-1} u_{n+[\epsilon n]-i} \frac{1}{\mathcal{H}(i)}. \tag{5.7}$$

By (4.23) we see that $\mathcal{H}(n)$ is slowly varying and therefore so is $1/\mathcal{H}(n)$. Consequently for any $\delta > 0$ small, and using (4.22) we see that for some $C < \infty$ independent of δ

$$\sum_{i=0}^{\delta n} u_{n+[\epsilon n]-i} \frac{1}{\mathcal{H}(i)} \leq c n^{-1} \sum_{i=0}^{\delta n} \frac{1}{\mathcal{H}(i)} \leq \frac{C\delta}{\mathcal{H}(n)}. \tag{5.8}$$

Using this and the slow variation of $1/\mathcal{H}(n)$ we find that

$$\sum_{i=0}^{n-1} u_{n+[\epsilon n]-i} \frac{1}{\mathcal{H}(i)} \sim \frac{1}{\mathcal{H}(n)} \sum_{i=0}^{n-1} u_{n+[\epsilon n]-i} \sim \frac{\mathcal{H}(n+[\epsilon n]) - \mathcal{H}([\epsilon n])}{\mathcal{H}(n)}. \tag{5.9}$$

Thus by (5.7)

$$r_n \geq \frac{1}{\mathcal{H}([\epsilon n])} - \frac{\mathcal{H}(n+[\epsilon n]) - \mathcal{H}([\epsilon n])}{\mathcal{H}([\epsilon n])\mathcal{H}(n)} + o\left(\frac{1}{\mathcal{H}^2(n)} \right). \tag{5.10}$$

By (4.24)

$$\frac{1}{\mathcal{H}([\epsilon n])} - \frac{1}{\mathcal{H}(n)} = \frac{1}{2\pi\sqrt{\det\Gamma}} \frac{\log \epsilon^{-1}}{\mathcal{H}^2(n)} + o\left(\frac{1}{\mathcal{H}^2(n)}\right). \tag{5.11}$$

Using (4.23) and (4.24) this shows that

$$\begin{aligned} r_n &\geq \frac{1}{\mathcal{H}(n)} + \frac{1}{2\pi\sqrt{\det\Gamma}} \frac{(\log \epsilon^{-1} - \log(1+\epsilon)\epsilon^{-1})}{\mathcal{H}^2(n)} + o\left(\frac{1}{\mathcal{H}^2(n)}\right) \\ &= \frac{1}{\mathcal{H}(n)} - \frac{1}{2\pi\sqrt{\det\Gamma}} \frac{\log(1+\epsilon)}{\mathcal{H}^2(n)} + o\left(\frac{1}{\mathcal{H}^2(n)}\right). \end{aligned} \tag{5.12}$$

Since $\epsilon > 0$ is arbitrary we have shown that

$$r_n = \frac{1}{\mathcal{H}(n)} + o\left(\frac{1}{\mathcal{H}^2(n)}\right). \tag{5.13}$$

Then by (5.3) and the slow variation of $1/\mathcal{H}(n)$

$$\mathbb{E}R_n = \sum_{i=0}^{n} \frac{1}{\mathcal{H}(i)} + o\left(\frac{n}{\mathcal{H}^2(n)}\right). \tag{5.14}$$

We have

$$\sum_{i=0}^{n} \frac{1}{\mathcal{H}(i)} = \frac{n+1}{\mathcal{H}(n)} + \sum_{i=0}^{n} \frac{\mathcal{H}(n) - \mathcal{H}(i)}{\mathcal{H}(i)\mathcal{H}(n)} \tag{5.15}$$

and using (4.22) and then the slow variation of $1/\mathcal{H}(n)$

(5.16)

$$\sum_{i=0}^{n} \frac{\mathcal{H}(n) - \mathcal{H}(i)}{\mathcal{H}(i)} = \sum_{j=1}^{n} u_j \sum_{i=0}^{j-1} \frac{1}{\mathcal{H}(i)} \sim \frac{1}{2\pi\sqrt{\det\Gamma}} \sum_{i=0}^{n} \frac{1}{\mathcal{H}(i)} \sim \frac{1}{2\pi\sqrt{\det\Gamma}} \frac{n}{\mathcal{H}(n)}$$

and together with (5.14) and (5.15) this completes the proof of (5.1). □

Proposition 5.2 *Suppose $\{X_i\}$ is a sequence of i.i.d. mean zero random vectors taking values in $\mathbb{Z}^2$ with*

$$\mathbb{E}\left(|X|^2 (\log^+ |X|)^{\frac{1}{2}+\delta}\right) < \infty \tag{5.17}$$

for some $\delta > 0$ *and nondegenerate covariance matrix* Γ. *Let* $S_n = \sum_{i=1}^n X_i$ *and suppose* S_n *is strongly aperiodic. Then*

$$\mathbb{P}(S_n = 0) = \frac{1}{2\pi n\sqrt{\det \Gamma}} + O\Big(\frac{1}{n(\log n)^{(1+\delta)/2}}\Big). \tag{5.18}$$

Proof. Let φ be the characteristic function of X_i, let $x \cdot y$ denote the inner product in $\mathbb{R}^2$, let $Q(u) = u \cdot \Gamma u$, and let $C = [-\pi, \pi]^2$. We observe that

$$\begin{aligned} &|1 - \varphi(u) - Q(u)| \\ &\quad = |\mathbb{E}\left(1 - e^{iu\cdot X} + iu \cdot X + (1/2)(iu \cdot X)^2\right)| \\ &\quad \le c_1 |u|^3 \mathbb{E}\left(1_{\{|X| \le 1/|u|\}} |X|^3\right) + c_1 |u|^2 \mathbb{E}\left(1_{\{|X| > 1/|u|\}} |X|^2\right) \end{aligned} \tag{5.19}$$

and consequently for any fixed $M > 0$

$$\begin{aligned} &|1 - \varphi(u/\sqrt{n}) - Q(u/\sqrt{n})| \\ &\quad \le c_2 \left(\frac{1}{n^{3/2}}\right) \mathbb{E}\left(1_{\{|u||X| \le \sqrt{n}\}} (|u||X|)^3\right) + c_2 \left(\frac{1}{n}\right) \mathbb{E}\left(1_{\{|u||X| > \sqrt{n}\}} (|u||X|)^2\right) \\ &\quad \le c_3 \frac{1}{n^{3/2}} + c_3 \left(\frac{1}{n^{3/2}}\right) \mathbb{E}\left(1_{\{M < |u||X| \le \sqrt{n}\}} (|u||X|)^3\right) \\ &\qquad\qquad + c_3 \left(\frac{1}{n}\right) \mathbb{E}\left(1_{\{|u||X| > \sqrt{n}\}} (|u||X|)^2\right). \end{aligned} \tag{5.20}$$

Choose M so that $x/(\log x)^{1/2+\delta}$ is monotone increasing on $x \ge M$, and therefore

$$\begin{aligned} &\mathbb{E}(1_{\{M < |u||X| \le \sqrt{n}\}} (|u||X|)^3) \\ &\quad \le \mathbb{E}\left(1_{\{M < |u||X| \le \sqrt{n}\}} (|u||X|)^2 (\log(|u||X|))^{1/2+\delta} \frac{|u||X|}{(\log(|u||X|))^{1/2+\delta}}\right) \\ &\quad \le \left(\frac{\sqrt{n}}{(\log(\sqrt{n}))^{1/2+\delta}}\right) \mathbb{E}\left(1_{\{M < |u||X| \le \sqrt{n}\}} (|u||X|)^2 (\log(|u||X|))^{1/2+\delta}\right). \end{aligned} \tag{5.21}$$

Also

$$\begin{aligned} &\mathbb{E}(1_{\{|u||X| > \sqrt{n}\}} (|u||X|)^2) \\ &\quad \le \left(\frac{1}{(\log(\sqrt{n}))^{1/2+\delta}}\right) \mathbb{E}\left(1_{\{|u||X| > \sqrt{n}\}} (|u||X|)^2 (\log(|u||X|))^{1/2+\delta}\right). \end{aligned} \tag{5.22}$$

(5.20) then implies that

$$|1-\varphi(u/\sqrt{n})-Q(u/\sqrt{n})| \le c\frac{|u|^2|(\log(|u|))^{1/2+\delta}|}{n(\log(n))^{1/2+\delta}}. \tag{5.23}$$

Following the proof in Spitzer [29], pp. 76–77,

$$\begin{aligned}
2\pi n\mathbb{P}(S_n=0) &= (2\pi)^{-1/2} n\int_C (\varphi(u))^n du \\
&= (2\pi)^{-1}\int_{\sqrt{n}C} \varphi(u/\sqrt{n})^n du \\
&= I_0 + I_1(n,A_n) + I_2(n,A_n) + I_3(n,A_n,r) + I_4(n,r),
\end{aligned}$$

where

$$\begin{aligned}
I_0 &= (2\pi)^{-1}\int_{\mathbb{R}^2} e^{-Q(u)/2} du = (\det Q)^{-1/2}, \\
I_1(n,A_n) &= (2\pi)^{-1}\int_{|u|\le A_n} [\varphi(u/\sqrt{n})^n - e^{-Q(u)/2}]\, du, \\
I_2(n,A_n) &= -(2\pi)^{-1}\int_{|u|>A_n} e^{-Q(u)/2} du, \\
I_3(n,A_n,r) &= (2\pi)^{-1}\int_{A_n<|u|<r\sqrt{n}} \varphi(u/\sqrt{n})^n\, du, \\
I_4(n,r) &= (2\pi)^{-1}\int_{|u|\ge r\sqrt{n},\, u\in\sqrt{n}C} \varphi(u/\sqrt{n})^n\, du.
\end{aligned}$$

Since X has second moments, a Taylor expansion shows that

$$\varphi(u) = 1 - \frac{Q(u)}{2} + o(|u|^2),$$

and hence we can choose r such that $|\varphi(u/\sqrt{n})^n| \le e^{-Q(u)/4}$ if $|u| \le r\sqrt{n}$. By the strong aperiodicity there exists $\gamma > 0$ such that $|\varphi(u/\sqrt{n})| \le 1-\gamma$ if $|u| > r\sqrt{n}$ and $u \in \sqrt{n}C$. Set $A_n = c_4\sqrt{\log\log n}$. We have

$$|I_4(n,r)| \le (2\pi)^{-1}\int_{u\in\sqrt{n}C} (1-\gamma)^n du = O(n^{-p})$$

for every positive integer p. Next

$$|I_3(n,A_n,r)| \le \int_{|u|>c_4\sqrt{\log\log n}} e^{-Q(u)/4} du = O((\log n)^{-2})$$

for c_4 large and similarly we have the same bound for $|I_2(n, A_n)|$. To estimate $I_1(n, A_n)$ we use the inequality $|a^n - b^n| \le n|a - b|$ if $|a|, |b| \le 1$ with $a = \varphi(u/\sqrt{n})$ and $b = e^{-Q(u)/2n}$. Using (5.23) and the analogous expansion for $e^{-Q(u)/2n}$ we have

$$\begin{aligned} |\varphi(u/\sqrt{n})^n - e^{-Q(u)/2}| &\le n|\varphi(u/\sqrt{n}) - e^{-Q(u)/2n}| \\ &\le c_5 n \frac{|u|^2 |(\log(|u|))^{1/2+\delta}|}{n(\log(n))^{1/2+\delta}} = c_5 \frac{|u|^2 |(\log(|u|))^{1/2+\delta}|}{(\log(n))^{1/2+\delta}}. \end{aligned}$$

Integrating this over the set $\{|u| \le A_n\}$, we see

$$|I_1(n, A_n)| = O((\log\log n)^{2+\delta/2}/(\log n)^{1/2+\delta}) = O(1/(\log n)^{(1+\delta)/2}).$$

Summing I_0 through I_4, we obtain

$$2\pi n \mathbb{P}(S_n = 0) = (\det \Gamma)^{-1/2} + O(1/(\log n)^{(1+\delta)/2}).$$

□

We now apply Theorem 4.1 to establish some sharp exponential estimates for the range and intersection of ranges. Aside from their intrinsic interest, they will be used to estimate the tail probabilities in our first main theorem.

We write $S(I)$ for $\{S_k : k \in I\}$. Let $S^{(i)}$, $i = 1, \ldots, p$ be p independent copies of S. Taking $f_n^x(S) = 1_{\{x \in S[1,n]\}}$ in Theorem 4.1 yields Corollary 1 of [8]: for any integers $a \ge 1$, $n_1, \cdots n_a \ge 1$,

$$\left(\mathbb{E} J^m_{n_1+\cdots+n_a}\right)^{1/p} \le \sum_{\substack{k_1+\cdots+k_a=m \\ k_1,\cdots,k_a \ge 0}} \frac{m!}{k_1! \cdots k_a!} \left(\mathbb{E} J^{k_1}_{n_1}\right)^{1/p} \cdots \left(\mathbb{E} J^{k_a}_{n_a}\right)^{1/p}, \tag{5.24}$$

where

$$J_n = \#\big\{S^{(1)}[1,n] \cap \cdots \cap S^{(p)}[1,n]\big\} \quad n = 1, 2, \cdots.$$

In the next Theorem we deduce from this the exponential integrability of J_n, which was established in [5] in the special case $p = 2$ and under the condition that S had bounded increments.

Theorem 5.3 *Assume that the planar random walk S has finite second moments and zero mean. There exists $\theta > 0$ such that*

$$\sup_n \sup_{y_1,\cdots,y_p} \mathbb{E}^{(y_1,\cdots,y_p)} \exp\left\{\theta \Big(\frac{(\log n)^p}{n}\Big)^{1/(p-1)} J_n^{1/(p-1)}\right\} < \infty. \tag{5.25}$$

Proof. The proof of (5.25) is a modification of the approach used in Lemma 1 of [8]. We begin by showing that there is a constant $C > 0$ such that

$$\sup_n \mathbb{E}J_n^m \le C^m (m!)^{p-1}\Big(\frac{n}{(\log n)^p}\Big)^m, \qquad m,n = 1,2,\cdots . \tag{5.26}$$

We first consider the case $m \le (\log n)^{(p-1)/p}$. Write $l(n,m) = [n/m] + 1$. Then by (5.24) and (4.38),

$$\begin{aligned}
\big(\mathbb{E}J_n^m\big)^{1/p} &\le \sum_{\substack{k_1+\cdots+k_m=m\\ k_1,\cdots,k_m\ge 0}} \frac{m!}{k_1!\cdots k_m!}\big(\mathbb{E}J_{l(n,m)}^{k_1}\big)^{1/p}\cdots\big(\mathbb{E}J_{l(n,m)}^{k_m}\big)^{1/p}\\
&\le \sum_{\substack{k_1+\cdots+k_m=m\\ k_1,\cdots,k_m\ge 0}} \frac{m!}{k_1!\cdots k_m!} k_1!\cdots k_m!\big(\mathbb{E}J_{l(n,m)}\big)^{k_1/p}\cdots\big(\mathbb{E}J_{l(n,m)}\big)^{k_m/p}\\
&= \binom{2m-1}{m} m!\big(\mathbb{E}J_{l(n,m)}\big)^{m/p} \le \binom{2m-1}{m} m! C^m\Big(\frac{(n/m)}{(\log n)^p}\Big)^{m/p}\\
&\le \binom{2m}{m}(m!)^{\frac{p-1}{p}} C^m\Big(\frac{n}{(\log n)^p}\Big)^{m/p},
\end{aligned}$$

where the second inequality follows from (4.42) and the third from (4.38) using the fact that $m = O(\log n)$ so that $\log n = O(\log(n/m))$. Hence, taking p-th powers we obtain

$$\mathbb{E}J_n^m \le \binom{2m}{m}^p C^{pm}(m!)^{p-1}\Big(\frac{n}{(\log n)^p}\Big)^m,$$

and (5.26) for the case of $m \le (\log n)^{(p-1)/p}$ follows from the fact

$$\binom{2m}{m} \le 4^m.$$

For the case $m > (\log n)^{(p-1)/p}$, notice from the definition of J_n that $J_n \le n$. So we have

$$\begin{aligned}
\mathbb{E}J_n^m \le n^m = (\log n)^{pm}\Big(\frac{n}{(\log n)^p}\Big)^m &\le m^{(p-1)m}\Big(\frac{n}{(\log n)^p}\Big)^m\\
&\le (m!)^{p-1}C^m\Big(\frac{n}{(\log n)^p}\Big)^m,
\end{aligned}$$

where the last step follows from Stirling's formula. This completes the proof of (5.26).

We next note that

$$\begin{aligned}
\mathbb{E}^{(y_1,\cdots,y_p)} J_n^m &= \sum_{x_1,\cdots,x_m\in\mathbb{Z}^2} \prod_{j=1}^p \mathbb{E}\prod_{k=1}^m 1_{\{x_k+y_j\in S[1,n]\}} \\
&\le \prod_{j=1}^p \bigg\{\sum_{x_1,\cdots,x_m\in\mathbb{Z}^2}\bigg[\mathbb{E}\prod_{k=1}^m 1_{\{x_k+y_j\in S[1,n]\}}\bigg]^p\bigg\}^{1/p} \\
&= \sum_{x_1,\cdots,x_m\in\mathbb{Z}^2}\bigg[\mathbb{E}\prod_{k=1}^m 1_{\{x_k\in S[1,n]\}}\bigg]^p = \mathbb{E}J_n^m
\end{aligned}$$

where the third step follows by translation invariance. Using Hölder's inequality we now see that

$$\begin{aligned}
(5.27)\qquad & \Big(\frac{(\log n)^p}{n}\Big)^{m/(p-1)} \sup_{y_1,\cdots,y_p} \mathbb{E}^{(y_1,\cdots,y_p)}\Big(J_n^{m/(p-1)}\Big) \\
&\le \Big(\frac{(\log n)^p}{n}\Big)^{m/(p-1)} \sup_{y_1,\cdots,y_p} \Big\{\mathbb{E}^{(y_1,\cdots,y_p)}\Big(J_n^m\Big)\Big\}^{1/(p-1)} \\
&\le \Big(\frac{(\log n)^p}{n}\Big)^{m/(p-1)} \Big\{\mathbb{E}\Big(J_n^m\Big)\Big\}^{1/(p-1)} \le C^m m!.
\end{aligned}$$

Our theorem then follows from a Taylor expansion. □

Remark. Theorem 5.3 is sharp in the sense that (5.25) does not hold if θ is too large. Indeed, by [24], for any $m=1,2,\cdots$,

$$\frac{(\log n)^{pm}}{n^m}\mathbb{E}J_n^m \longrightarrow (2\pi)^{pm}\det(\Gamma)^{m/2}\mathbb{E}\alpha\big([0,1]^p\big)^m$$

as $n\to\infty$, where $\alpha\big([0,1]^p\big)$ is the Brownian intersection local time formally defined by

$$\alpha\big([0,1]^p\big) = \int_{\mathbb{R}^d}\bigg[\prod_{j=1}^p\int_0^1 \delta_x\big(W_j(s)\big)ds\bigg]dx,$$

and by Theorem 2.1 in [7]

$$\mathbb{E}\exp\Big\{\theta\alpha\big([0,1]^p\big)^{(p-1)^{-1}}\Big\} = \infty$$

for large θ. The following theorem is sharp in the same sense.

Theorem 5.4 *Assume that the planar random walk S has finite second moments and zero mean. Then there exists $\theta > 0$ such that*

$$\sup_n \mathbb{E} \exp\Big\{\theta \frac{(\log n)^2}{n} |\overline{R}_n|\Big\} < \infty. \tag{5.28}$$

Proof. We first consider the case where n is replaced by 2^n. Let

$$N = [2(\log 2)^{-1} \log n]$$

so that $2^N \sim n^2$ and note that

$$\#\big\{S[1, 2^n]\big\} = \sum_{k=1}^{2^N} \#\big\{S((k-1)2^{n-N}, k2^{n-N}]\big\} \tag{5.29}$$
$$- \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} \#\Big\{S\big((2k-2)2^{n-j}, (2k-1)2^{n-j}\big] \cap S\big((2k-1)2^{n-j}, (2k)2^{n-j}\big]\Big\}.$$

Setting

$$\beta_k = \#\big\{S((k-1)2^{n-N}, k2^{n-N}]\big\}$$

and

$$\alpha_{j,k} = \#\Big\{S\big((2k-2)2^{n-j}, (2k-1)2^{n-j}\big] \cap S\big((2k-1)2^{n-j}, (2k)2^{n-j}\big]\Big\}$$

leads to the decomposition

$$\overline{R}_{2^n} = \sum_{k=1}^{2^N} \overline{\beta}_k - \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} \overline{\alpha}_{j,k}.$$

We will need the following lemma which is [8, Lemma 3].

Lemma 5.5

$$\sup_n \mathbb{E} \exp\Big\{\lambda \frac{\log n}{n} \#\big\{S[1, n]\big\}\Big\} < \infty \tag{5.30}$$

for all $\lambda > 0$.

Proof of Lemma 5.5: We first claim that for any $a, b > 0$ and any integer $n \geq 1$

$$\mathbb{P}\Big\{\#\{S[1,n]\} \geq a+b\Big\} \leq \mathbb{P}\Big\{\#\{S[1,n]\} \geq a\Big\}\mathbb{P}\Big\{\#\{S[1,n]\} \geq b\Big\}.$$

Notice that $\#\{S[1,n]\}$ takes integer values. So we may assume that a and b are integers for otherwise we can use $[a]$, $[b]$ and $[a+b]$ instead. Define

$$\tau = \inf\{k \geq 1; \ \ \#\{S[1,k]\} \geq a\}.$$

Then

$$\begin{aligned}
&\mathbb{P}\Big\{\#\{S[1,n]\} \geq a+b\Big\} = \mathbb{P}\Big\{\#\{S[1,n]\} \geq a+b, \ \ \tau \leq n\Big\} \\
&= \sum_{k=1}^n \mathbb{P}\Big\{\#\{S[1,n]\} - \#\{S[1,k]\} \geq b, \ \ \tau = k\Big\} \\
&\leq \sum_{k=1}^n \mathbb{P}\Big\{\#\{S[k+1,n]\} \geq b, \ \ \tau = k\Big\} \\
&= \sum_{k=1}^n \mathbb{P}\{\tau = k\}\mathbb{P}\Big\{\#\{S[1,n-k]\} \geq b\Big\} \\
&\leq \mathbb{P}\Big\{\#\{S[1,n]\} \geq b\Big\}.
\end{aligned}$$

To prove (5.30), let $C > 0$ be a constant such that

$$\mathbb{P}\Big\{\#\{S[1,n]\} \geq C\frac{n}{\log n}\Big\} \leq e^{-2}$$

for all $n \geq 1$. We have that

$$\mathbb{P}\Big\{\#\{S[1,n]\} \geq Cm\frac{n}{\log n}\Big\} \leq \bigg(\mathbb{P}\Big\{\#\{S[1,n]\} \geq C\frac{n}{\log n}\Big\}\bigg)^m \leq e^{-2m}$$

for all $n \geq 1$. Consequently,

$$\sup_n \mathbb{E}\exp\Big\{\lambda\frac{\log n}{n}\#\{S[1,n]\}\Big\} < \infty$$

for $\lambda \leq C^{-1}$. For $\lambda > C^{-1}$, one can take $\delta > 0$ such that $\lambda < C^{-1}[\delta^{-1}]$.

Write $k_n = [\delta]$. By subadditivity

$$\begin{aligned} &\mathbb{E}\exp\Big\{\lambda\frac{\log n}{n}\#\{S[1,n]\}\Big\} \\ &\le \bigg(\mathbb{E}\exp\Big\{\lambda\frac{\log n}{n}\#\{S[1,k_n]\}\Big\}\bigg)^{[\delta^{-1}]+1} \\ &\le \bigg(\mathbb{E}\exp\Big\{C^{-1}\frac{\log k_n}{k_n}\#\{S[1,k_n]\}\Big\}\bigg)^{[\delta^{-1}]+1}. \end{aligned}$$

□

In particular, it follows from (5.30) that

$$\sup_n \mathbb{E}\exp\Big\{\lambda\frac{\log 2^{n-N}}{2^{n-N}}|\overline{\beta}_1|\Big\} < \infty.$$

Notice that $\overline{\beta}_1, \cdots, \overline{\beta}_{2^N}$ is an i.i.d. sequence with $\mathbb{E}\overline{\beta}_1 = 0$. By Lemma 4.4, there is a $\theta > 0$ such that

$$\sup_n \mathbb{E}\exp\Big\{\theta 2^{-N/2}\frac{\log 2^{n-N}}{2^{n-N}}\Big|\sum_{k=1}^{2^N}\overline{\beta}_k\Big|\Big\} < \infty.$$

By the choice of N one can see that there is a $c > 0$ independent of n such that

$$2^{-N/2}\frac{\log 2^{n-N}}{2^{n-N}} \ge c\frac{(\log 2^n)^2}{2^n}.$$

So there is some $\theta > 0$ such that

$$\sup_n \mathbb{E}\exp\Big\{\theta\frac{(\log 2^n)^2}{2^n}\Big|\sum_{k=1}^{2^N}\overline{\beta}_k\Big|\Big\} < \infty.$$

We need to show that for some $\theta > 0$,

$$\sup_n \mathbb{E}\exp\Big\{\theta\frac{(\log 2^n)^2}{2^n}\Big|\sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}}\overline{\alpha}_{j,k}\Big|\Big\} < \infty. \tag{5.31}$$

Set

$$\tilde{J}_n = \#\big\{S[1,n]\cap S'[1,n]\big\} \qquad n = 1, 2, \cdots, \tag{5.32}$$

where S' is an independent copy of the random walk S. In our notation, for each $1 \le j \le N$, $\{\overline{\alpha}_{j,1}, \cdots, \overline{\alpha}_{j,2^{j-1}}\}$ is an i.i.d. sequence with the same distribution as $\tilde{J}_{2^{n-j}}$. By Theorem 5.3 (with $p = 2$), there is a $\delta > 0$ such that

$$\sup_n \sup_{j \le N} \mathbb{E} \exp\left\{\delta \frac{(\log 2^{n-j})^2}{2^{n-j}} \big|\overline{\alpha}_{j,1}\big|\right\} < \infty.$$

By Lemma 4.4 again, there is a $\bar{\theta} > 0$ such that

$$\sup_n \sup_{j \le N} \mathbb{E} \exp\left\{\bar{\theta} 2^{-j/2} \frac{(\log 2^n)^2}{2^{n-j}} \Big| \sum_{k=1}^{2^{j-1}} \overline{\alpha}_{j,k}\Big|\right\} < \infty.$$

Hence for some $\theta > 0$

$$C(\theta) \equiv \sup_n \sup_{j \le N} \mathbb{E} \exp\left\{\theta 2^{j/2} \frac{(\log 2^n)^2}{2^n} \Big| \sum_{k=1}^{2^{j-1}} \overline{\alpha}_{j,k}\Big|\right\} < \infty.$$

Write

$$\lambda_N = \prod_{j=1}^{N} \big(1 - 2^{-j/2}\big) \quad \text{and} \quad \lambda_\infty = \prod_{j=1}^{\infty} \big(1 - 2^{-j/2}\big).$$

Using Hölder's inequality with $1/p = 1 - 2^{-N/2}$, $1/q = 2^{-N/2}$ we have

$$\begin{aligned}
&\mathbb{E} \exp\left\{\lambda_N \theta \frac{(\log 2^n)^2}{2^n} \Big| \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} \overline{\alpha}_{j,k}\Big|\right\} \\
&\qquad \le \left(\mathbb{E} \exp\left\{\lambda_{N-1} \theta \frac{(\log 2^n)^2}{2^n} \Big| \sum_{j=1}^{N-1} \sum_{k=1}^{2^{j-1}} \overline{\alpha}_{j,k}\Big|\right\}\right)^{1-2^{-N/2}} \\
&\qquad\quad \times \left(\mathbb{E} \exp\left\{\lambda_N \theta 2^{N/2} \frac{(\log 2^n)^2}{2^n} \Big| \sum_{k=1}^{2^{N-1}} \overline{\alpha}_{N,k}\Big|\right\}\right)^{2^{-N/2}} \\
&\qquad \le \mathbb{E} \exp\left\{\lambda_{N-1} \theta \frac{(\log 2^n)^2}{2^n} \Big| \sum_{j=1}^{N-1} \sum_{k=1}^{2^{j-1}} \overline{\alpha}_{j,k}\Big|\right\} \cdot C(\theta)^{2^{-N/2}}
\end{aligned}$$

since $\lambda_N < 1$. Repeating this procedure,

$$\begin{aligned}
&\mathbb{E} \exp\left\{\lambda_N \theta \frac{(\log 2^n)^2}{2^n} \Big| \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} \overline{\alpha}_{j,k}\Big|\right\} \\
&\qquad \le C(\theta)^{2^{-1/2} + \cdots + 2^{-N/2}} \le C(\theta)^{2^{-1/2}(1-2^{-1/2})^{-1}}.
\end{aligned}$$

So we have

$$\sup_n \mathbb{E}\exp\left\{\lambda_\infty\theta\frac{(\log 2^n)^2}{2^n}\Big|\sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}}\overline{\alpha}_{j,k}\Big|\right\}\le C(\theta)^{2^{-1/2}(1-2^{-1/2})^{-1}}.$$

We have proved (5.31) and therefore (5.28) when n is the power of 2. We now prove Theorem 5.4 for general n. Given an integer $n\ge 2$, we have the following unique representation:

$$n=2^{m_1}+2^{m_2}+\cdots+2^{m_l}$$

where $m_1>m_2>\cdots m_l\ge 0$ are integers. Write

$$n_0=0 \quad\text{and}\quad n_i=2^{m_1}+\cdots+2^{m_i}\qquad i=1,\cdots,l.$$

Then

$$\begin{aligned}\#\big\{S[1,n]\big\}&=\sum_{i=1}^{l}\#\big\{S(n_{i-1},n_i]\big\}-\sum_{i=1}^{l-1}\#\big\{S(n_{i-1},n_i]\cap S(n_i,n]\big\}\\&=\sum_{i=1}^{l}B_i-\sum_{i=1}^{l-1}A_i.\end{aligned}$$

Write

$$\sum_{i=1}^{l}B_i=\sum_i{}'B_i+\sum_i{}''B_i$$

where $\sum_i{}'$ is the summation over i with $2^{m_i}\ge\sqrt{n}$ and $\sum_i{}''$ is the summation over i with $2^{m_i}<\sqrt{n}$. We also define the products $\prod_i{}'$ and $\prod_i{}''$ in a similar manner. Then

$$\begin{aligned}&\mathbb{E}\exp\left\{\theta\frac{(\log n)^2}{n}\Big|\sum_i{}'(B_i-\mathbb{E}B_i)\Big|\right\}\\&\qquad\le\prod_i{}'\left(\mathbb{E}\exp\left\{\theta\frac{(\log n)^2}{n}2^{-m_i}\Big(\sum_j{}'2^{m_j}\Big)|\overline{R}_{2^{m_i}}|\right\}\right)^{2^{m_i}\left(\sum_j{}'2^{m_j}\right)^{-1}}\\&\qquad\le\prod_i{}'\left(\mathbb{E}\exp\left\{4\theta\frac{(\log 2^{m_i})^2}{2^{m_i}}|\overline{R}_{2^{m_i}}|\right\}\right)^{2^{m_i}\left(\sum_j{}'2^{m_j}\right)^{-1}}\\&\qquad\le\sup_m\mathbb{E}\exp\left\{4\theta\frac{(\log 2^m)^2}{2^m}|\overline{R}_{2^m}|\right\}.\end{aligned}$$

Assume that the set $\{1 \le i \le l;\ 2^{m_i} < \sqrt{n}\}$ is non-empty. We have

$$\sum_i{}'' 2^{m_i} \le 2\sqrt{n}.$$

So we have

$$\frac{(\log n)^2}{n} \le \frac{1}{\sqrt{n}} \le 2\Big(\sum_i{}'' 2^{m_i}\Big)^{-1}.$$

Hence

$$\begin{aligned}
&\mathbb{E}\exp\Big\{\theta\frac{(\log n)^2}{n}\Big|\sum_i{}''(B_i - \mathbb{E}B_i)\Big|\Big\} \\
&\quad \le \prod_i{}''\bigg(\mathbb{E}\exp\Big\{\theta\frac{(\log n)^2}{n}2^{-m_i}\Big(\sum_j{}''2^{m_j}\Big)|\overline{R}_{2^{m_i}}|\Big\}\bigg)^{2^{m_i}\left(\sum_j{}''2^{m_j}\right)^{-1}} \\
&\quad \le \prod_i{}''\bigg(\mathbb{E}\exp\Big\{2\theta\frac{1}{2^{m_i}}|\overline{R}_{2^{m_i}}|\Big\}\bigg)^{2^{m_i}\left(\sum_j{}''2^{m_j}\right)^{-1}} \\
&\quad \le \sup_m \mathbb{E}\exp\Big\{2\theta\frac{1}{2^m}|\overline{R}_{2^m}|\Big\}.
\end{aligned}$$

By the Cauchy-Schwarz inequality and what we have proved in the previous step, there exists $\theta > 0$ such that

$$\mathbb{E}\exp\Big\{\theta\frac{(\log n)^2}{n}\Big|\sum_{i=1}^{l}(B_i - \mathbb{E}B_i)\Big|\Big\}$$

is bounded uniformly in n. By the fact that

$$n - n_i = 2^{m_{i+1}} + \cdots + 2^{m_l} \le 2^{m_i} \tag{5.33}$$

we have

$$A_i \overset{d}{=} \#\big\{S[1, 2^{m_i}] \cap S'[1, n - n_i]\big\} \le J_{2^{m_i}}. \tag{5.34}$$

By (4.38) there is a constant $C > 0$ independent of n such that

$$\sum_{i=1}^{l-1} \mathbb{E}A_i \le \sum_{i=1}^{l-1} \mathbb{E}J_{2^{m_i}} \le C\sum_{i=1}^{l} \frac{2^{m_i}}{m_i^2} \tag{5.35}$$

$$\le C\sum_{m_i<l/2} \frac{2^{m_i}}{m_i^2} + C\sum_{m_i\ge l/2}^{l} \frac{2^{m_i}}{m_i^2}$$

$$\le C2^{l/2} + C\frac{n}{(\log n)^2}$$

$$\le C\frac{n}{(\log n)^2}.$$

It remains to show that

$$\sup_n \mathbb{E}\exp\left\{\theta\frac{(\log n)^2}{n}\sum_{i=1}^{l-1} A_i\right\} < \infty. \tag{5.36}$$

Using (5.35) this follows from (5.25), (with $p = 2$), and the same argument used for $B_1 - \mathbb{E}B_1, \cdots, B_l - \mathbb{E}B_l$. □

In view of the remark prior to Theorem 5.4, the next result shows that $\overline{R}_n$ has a non-symmetric tail behavior.

Theorem 5.6 *Under the assumptions of Theorem 5.4,*

$$\sup_n \mathbb{E}\exp\left\{\theta\frac{(\log n)^2}{n}\overline{R}_n\right\} < \infty \tag{5.37}$$

for all $\theta > 0$.

Proof. By Theorem 5.4, (5.37) holds for some $\theta_0 > 0$. For $\theta > \theta_0$, take an integer $m \ge 1$ such that $m^{-1}\theta < \theta_0$. It is easy to see that it suffices to prove

$$\sup_n \mathbb{E}\exp\left\{\theta\frac{(\log n)^2}{mn}\overline{R}_{nm}\right\} < \infty. \tag{5.38}$$

Set $\zeta_{jn} = \#\{S((j-1)n, jn]\}$. By the facts that

$$\overline{R}_{nm} \le \sum_{j=1}^{m} \overline{\zeta_{jn}} + \Big(\sum_{j=1}^{m} \mathbb{E}\zeta_{jn}\Big) - \mathbb{E}R_{nm}$$

and that by (4.23) and (4.24),

$$\begin{aligned}\Big(\sum_{j=1}^{m}\mathbb{E}\zeta_{jn}\Big)-\mathbb{E}R_{mn} &= m\mathbb{E}R_n-\mathbb{E}R_{mn}\\ &= \frac{mn}{\mathcal{H}(n)}+O\Big(\frac{mn}{\mathcal{H}(n)^2}\Big)-\frac{mn}{\mathcal{H}(mn)}+O\Big(\frac{mn}{\mathcal{H}(mn)^2}\Big)\\ &= \frac{mn}{\mathcal{H}(n)\mathcal{H}(mn)}(\mathcal{H}(mn)-\mathcal{H}(n))+O\Big(\frac{n}{(\log n)^2}\Big)\\ &= O\Big(\frac{n}{(\log n)^2}\Big)\end{aligned}$$

as $n\to\infty$ (note m is fixed), there is a constant $C_{m,\theta}>0$ depending only on m and θ such that

$$\mathbb{E}\exp\Big\{\theta\frac{(\log n)^2}{mn}\overline{R}_{nm}\Big\}\leq C_m\Big(\mathbb{E}\exp\Big\{\theta\frac{(\log n)^2}{mn}\overline{R}_n\Big\}\Big)^m.$$

So we have (5.38). □

6 Moderate deviations for $R_n-\mathbb{E}R_n$

We can now prove Theorem 1.1.

Proof. We first prove the upper bound. Let $t>0$ and write $K=[t^{-1}b_n]$. Divide $[1,n]$ into K disjoint subintervals, each of length $[n/K]$ or $[n/K]+1$. Call the i^{th} subinterval I_i. Let $E_i=\#\{S(I_i)\}$. Then

$$\overline{R}_n\leq\sum_{j=1}^{K}\overline{E}_j+\Big(\sum_{j=1}^{K}\mathbb{E}E_j\Big)-\mathbb{E}R_n$$

From (5.1) we have

(6.1)

$$\sum_{j=1}^{K}\mathbb{E}E_j-\mathbb{E}R_n$$
$$=K\frac{n/K}{\mathcal{H}([n/K])}-\frac{n}{\mathcal{H}(n)}+\frac{1}{2\pi\sqrt{\det\Gamma}}\Big\{K\frac{n/K}{\mathcal{H}^2([n/K])}-\frac{n}{\mathcal{H}^2(n)}\Big\}+o\Big(\frac{n}{\mathcal{H}^2(n)}\Big)$$
$$=\frac{n(\mathcal{H}(n)-\mathcal{H}([n/K]))}{\mathcal{H}^2(n)}\Big\{1+\frac{\mathcal{H}(n)-\mathcal{H}([n/K])}{\mathcal{H}([n/K])}\Big\}$$
$$+\frac{n}{\mathcal{H}^2(n)}\Big\{\frac{\mathcal{H}^2(n)-\mathcal{H}^2([n/K])}{\mathcal{H}^2([n/K])}\Big\}+o\Big(\frac{n}{\mathcal{H}^2(n)}\Big),$$

where the error term can be taken to be independent of $\{b_n\}$. (This is where the hypothesis $\log b_n=o((\log n)^{1/2})$ is used.) Since

$$\mathcal{H}(n)-\mathcal{H}([n/K])=\sum_{k=[n/K]+1}^{n}\mathbb{P}\{S_k=0\}\sim\frac{\log K}{2\pi\sqrt{\det\Gamma}},$$

we have

$$\sum_{j=1}^{K}\mathbb{E}E_j-\mathbb{E}R_n=\frac{n(\mathcal{H}(n)-\mathcal{H}([n/K]))}{\mathcal{H}^2(n)}+o\Big(\frac{n}{\mathcal{H}^2(n)}\Big). \tag{6.2}$$

Hence for any $\lambda>0$,

$$\mathbb{P}\Big\{\overline{R}_n\geq\frac{n}{\mathcal{H}^2(n)}\big(\mathcal{H}(n)-\mathcal{H}([n/b_n])\big)\Big\}$$
$$\leq\exp\Big\{-\lambda b_n\big(\mathcal{H}(n)-\mathcal{H}([n/b_n])\big)\Big\}\mathbb{E}\exp\Big\{\lambda\frac{\mathcal{H}^2(n)b_n}{n}\overline{R}_n\Big\}$$
$$\leq\exp\Big\{-\lambda b_n\big(\mathcal{H}([n/K])-\mathcal{H}([n/b_n])\big)+o(b_n)\Big\}$$
$$\times\Bigg(\mathbb{E}\exp\Big\{\lambda\frac{\mathcal{H}^2(n)b_n}{n}\overline{E}_1\Big\}\Bigg)^K.$$

Notice that

$$\lim_{n\to\infty}\big(\mathcal{H}([n/K])-\mathcal{H}([n/b_n])\big)=\frac{\log t}{2\pi\sqrt{\det\Gamma}}$$

and that by [24, Theorem 6.1] or (2.2),

$$\frac{\mathcal{H}^2(n)b_n}{n}\overline{E}_1 \xrightarrow{d} -\frac{2\pi t}{2\pi\sqrt{\det\Gamma}}\gamma_1,$$

where γ_t is the renormalized self-intersection local time of a planar Brownian motion. By Theorem 5.6 and the dominated convergence theorem,

$$\mathbb{E}\exp\Big\{\lambda\frac{\mathcal{H}^2(n)b_n}{n}\overline{E}_1\Big\} \longrightarrow \mathbb{E}\exp\Big\{-\lambda\frac{2\pi t}{2\pi\sqrt{\det\Gamma}}\gamma_1\Big\}$$

Consequently,

(6.3)

$$\begin{aligned}
&\limsup_{n\to\infty} b_n^{-1}\log\mathbb{P}\Big\{\overline{R}_n \geq \frac{n}{\mathcal{H}^2(n)}\big(\mathcal{H}(n)-\mathcal{H}([n/b_n])\big)\Big\}\\
&\qquad\leq -\lambda\frac{\log t}{2\pi\sqrt{\det\Gamma}} + \frac{1}{t}\log\mathbb{E}\exp\Big\{-\lambda\frac{2\pi t}{2\pi\sqrt{\det\Gamma}}\gamma_1\Big\}\\
&\qquad= \frac{\lambda}{2\pi\sqrt{\det\Gamma}}\log\frac{\lambda}{2\pi\sqrt{\det\Gamma}}\\
&\qquad\quad+\frac{1}{t}\log\mathbb{E}\exp\Big\{-\frac{\lambda t}{2\pi\sqrt{\det\Gamma}}\log\frac{\lambda t}{2\pi\sqrt{\det\Gamma}} - \lambda\frac{2\pi t}{2\pi\sqrt{\det\Gamma}}\gamma_1\Big\}.
\end{aligned}$$

The limit

$$C \equiv \lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\Big\{-t\log t - 2\pi t\gamma_1\Big\} \tag{6.4}$$

exists. Indeed, for any $s, t > 0$,

$$\begin{aligned}
\gamma_{s+t} &= \gamma_t + \gamma_s' + \int_0^t\int_t^{s+t}\delta_0(W_u - W_v)du\,dv - \mathbb{E}\int_0^t\int_t^{s+t}\delta_0(W_u - W_v)dv\,du\\
&\geq \gamma_t + \gamma_s' - \mathbb{E}\int_0^t\int_t^{s+t}\delta_0(W_u - W_v)du\,dv
\end{aligned}$$

where

$$\gamma_s' = \iint_{t\leq u<v\leq s+t}\delta_0(W_u - W_v)du\,dv - \mathbb{E}\iint_{t\leq u<v\leq s+t}\delta_0(W_u - W_v)du\,dv$$

has the same distribution as γ_s and is independent of γ_t. Notice that

$$\mathbb{E}\int_0^t\int_t^{s+t}\delta_0(W_u-W_v)dv\,du=\int_0^t\int_t^{s+t}\frac{1}{2\pi}\frac{1}{v-u}dv\,du$$
$$=\frac{1}{2\pi}\big[(s+t)\log(s+t)-s\log s-t\log t\big]$$

Summarizing what we have,

$$\mathbb{E}\exp\Big\{-(s+t)\log(s+t)-2\pi\gamma_{s+t}\Big\}$$
$$\le\mathbb{E}\exp\Big\{-s\log s-2\pi\gamma_s\Big\}\mathbb{E}\exp\Big\{-t\log t-2\pi\gamma_t\Big\}.$$

Therefore, the limit

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\Big\{-t\log t-2\pi\gamma_t\Big\}$$

exists. The existence of (6.4) follows since by scaling $\gamma_t\overset{d}{=}t\gamma_1$.

Set

$$L=\exp(-1-C).$$

Letting $t\to\infty$ in (6.3) gives

$$\limsup_{n\to\infty}b_n^{-1}\log\mathbb{P}\Big\{\overline{R}_n\ge\frac{n}{\mathcal{H}^2(n)}\big(\mathcal{H}(n)-\mathcal{H}([n/b_n])\big)\Big\}$$
$$\le\frac{\lambda}{2\pi\sqrt{\det\Gamma}}\log\frac{\lambda}{2\pi\sqrt{\det\Gamma}}+C\frac{\lambda}{2\pi\sqrt{\det\Gamma}}.$$

Taking

$$\frac{\lambda}{2\pi\sqrt{\det\Gamma}}=\exp\big\{-1-C\big\}$$

then yields

$$\limsup_{n\to\infty}b_n^{-1}\log\mathbb{P}\Big\{\overline{R}_n\ge\frac{n}{\mathcal{H}^2(n)}\big(\mathcal{H}(n)-\mathcal{H}([n/b_n])\big)\Big\}$$
$$\le-\exp\big\{-1-C\big\}=-L.$$

We now prove the lower bound. The proof is similar to that of Proposition 4.4 of [5]. Fix n and let $K=[b_n]$. Let $M=[n/b_n]$. Let I_j be the interval

$(m_j, m_{j+1}]$, where the m_j are integers such that $m_0 = 0$, $m_K = n$, and $m_{j+1} - m_j$ is equal to either M or $M+1$.

Let $\mathbf{e}$ be a vector of length $\sqrt{M}$ and let $B(x,r)$ be the ball of radius r about x. Set

$$E_j = \#\{S(I_j)\}, \qquad H_j = \#\{S(I_j) \cap S(I_{j-1})\}.$$

Let

$$A_j = \{S_{m_{j+1}} \in B((j+1)\mathbf{e}, \tfrac{1}{8}\sqrt{M})\} \cap \{S(I_j) \subset B((j+\tfrac{1}{2})\mathbf{e}, \sqrt{M})\} \tag{6.5}$$

and

$$B_j = \{\overline{E}_j (\log M)^2/M \geq -c_1\} \tag{6.6}$$

where we will select c_1 in a moment. By the central limit theorem, we know $\mathbb{P}^{S_{m_{j-1}}}(A_j) \geq c_2$ on the event A_{j-1} if n is large. By [24, Theorem 6.1] or (2.2), $\mathbb{P}^{S_{m_{j-1}}}(A_j \cap B_j) > c_2/2$ on the event A_{j-1} if we take c_1 sufficiently large. If we let

$$F = \bigcap_{j=0}^{K-1} (A_j \cap B_j),$$

then by the Markov property applied $K-1$ times we have

$$\mathbb{P}(F) \geq (c_2/2)^{K-1}. \tag{6.7}$$

On the set F we have that $S(I_j)$ is disjoint from $S(I_i)$ if $|i-j|>1$, and so on F

$$\overline{R}_n = \sum_{j=1}^K \overline{E}_j + \Big(\Big(\sum_{j=1}^K \mathbb{E}E_j\Big) - \mathbb{E}R_n\Big) - \sum_{j=1}^K H_j. \tag{6.8}$$

On the set F the event B_j holds for each j, and so

$$\sum_{j=1}^K \overline{E}_j \geq -\frac{c_1 KM}{(\log M)^2} \geq -\frac{c_3 n}{(\log n)^2}. \tag{6.9}$$

As in (6.2),

$$\Big(\sum_{j=1}^K \mathbb{E}E_j\Big) - \mathbb{E}R_n = \frac{n(\mathcal{H}(n) - \mathcal{H}([n/K]))}{\mathcal{H}(n)^2} + o\Big(\frac{n}{\mathcal{H}(n)^2}\Big) \tag{6.10}$$

if n is large.

Let $\Lambda > 0$ be chosen in a moment. Let

$$C_1 = \left\{ \sum_{\{j \ odd\}} H_j \geq \frac{n\Lambda}{(\log n)^2} \right\}, \qquad C_2 = \left\{ \sum_{\{j \ even\}} H_j \geq \frac{n\Lambda}{(\log n)^2} \right\}.$$

Set $G = F \cap C_1^c \cap C_2^c$. For j odd the H_j are independent, and by Theorem 5.3 with $p = 2$

$$\begin{aligned}\mathbb{P}(C_1) &= \mathbb{P}\Big(\sum_{\{j \ odd\}} \frac{H_j}{M/(\log M)^2} \geq c_4 K\Lambda \Big) \\ &\leq e^{-c_4c_5K\Lambda} \mathbb{E} e^{c_5(\sum H_j)(\log M)^2/M} \\ &\leq e^{-c_4c_5K\Lambda} c_6^K,\end{aligned}$$

where c_4, c_5, c_6 do not depend on Λ and without loss of generality we may assume $c_6 > 1$. Choose Λ large so that $e^{-c_4c_5\Lambda} \leq c_6^{-2}$. When n is large, K will be large, and then $\mathbb{P}(C_1) \leq \mathbb{P}(F)/3$. We have a similar estimate for $\mathbb{P}(C_2)$, so

$$\mathbb{P}(G) \geq (c_2/2)^{K-1}/3.$$

Set $v_n = \mathcal{H}(n) - \mathcal{H}([n/b_n])$. On the event G

$$\sum_{j=1}^{K} H_j \leq 2\frac{n\Lambda}{(\log n)^2}, \tag{6.11}$$

and so combining (6.9), (6.10), and (6.11), on the event G

$$\overline{R}_n \geq \Big(1 - \frac{c_7}{v_n}\Big) n v_n / \mathcal{H}(n)^2. \tag{6.12}$$

Therefore

$$\mathbb{P}\Big(\overline{R}_n \geq \Big(1 - \frac{c_7}{v_n}\Big) n v_n / \mathcal{H}(n)^2\Big) \geq c_8 c_9^{b_n}. \tag{6.13}$$

Define b_n' by $v_n' = \mathcal{H}(n) - \mathcal{H}([n/b_n']) = v_n + c_7$. If we apply (6.13) with b_n replaced by b_n', we have

$$\begin{aligned}\mathbb{P}(\overline{R}_n \geq & n v_n / \mathcal{H}(n)^2) \\ &= \mathbb{P}\Big(\overline{R}_n \geq \Big(1 - \frac{c_7}{v_n'}\Big) n v_n' / \mathcal{H}(n)^2\Big) \\ &\geq c_8 c_9^{b_n'}.\end{aligned}$$

We now take the logarithms of both sides, divide by b_n, and use the fact that the ratio b_n/b'_n is bounded above and below by positive constants to obtain the lower bound. □

Proof of Corollary 1.3: Assume first that S_n is strongly aperiodic. We have by Proposition 5.2 that

$$\mathbb{P}(S_n = 0) = \frac{1}{2\pi n\sqrt{\det\Gamma}} + O\Big(\frac{1}{n(\log n)^{1/2}}\Big). \tag{6.14}$$

Then, if γ denotes Euler's constant

$$\sum_{k=1}^{n} \frac{1}{k} = \log n + \gamma + O\Big(\frac{1}{n}\Big) \tag{6.15}$$

and

$$\sum_{k=3}^{n} \frac{1}{k(\log k)^{1/2}} \le \int_2^n \frac{dx}{x(\log x)^{1/2}} \le c_1(\log n)^{1/2} \tag{6.16}$$

so that

$$\begin{aligned} \mathcal{H}(n) &= \sum_{k=0}^{n} \mathbb{P}^0(S_k = 0) = 1 + \frac{1}{2\pi\sqrt{\det\Gamma}} \sum_{k=1}^{n} \left(\frac{1}{k} + O\Big(\frac{1}{k(\log k)^{1/2}}\Big)\right) \\ &= \frac{1}{2\pi\sqrt{\det\Gamma}} \left(\log n + \gamma + O\Big((\log n)^{1/2}\Big)\right) \\ &= \frac{\log n}{2\pi\sqrt{\det\Gamma}} \left(1 + O\Big(\frac{1}{(\log n)^{1/2}}\Big)\right). \end{aligned} \tag{6.17}$$

Similarly

$$\sum_{k=[n/b_n]+1}^{n} \frac{1}{k(\log k)^{1/2}} \le c_2\left((\log n)^{1/2} - (\log(n/b_n))^{1/2}\right). \tag{6.18}$$

To evaluate this note that

$$\begin{aligned} (\log(n/b_n))^{1/2} &= (\log n - \log b_n)^{1/2} \\ &= (\log n)^{1/2}(1 - \log b_n/\log n)^{1/2} \\ &= (\log n)^{1/2}(1 + O(\log b_n/\log n)) \\ &= (\log n)^{1/2} + O(\log b_n/(\log n)^{1/2}) \end{aligned} \tag{6.19}$$

by our assumption that $\log b_n = o((\log n)^{1/2})$. It follows that

$$\mathcal{H}(n) - \mathcal{H}([n/b_n]) = \frac{1}{2\pi\sqrt{\det\Gamma}} \sum_{k=[n/b_n]+1}^{n} \left(\frac{1}{k} + O\Big(\frac{1}{k(\log k)^{1/2}}\Big)\right) \tag{6.20}$$
$$= \frac{1}{2\pi\sqrt{\det\Gamma}} \left(\log b_n + O\Big(\frac{\log b_n}{(\log n)^{1/2}}\Big)\right)$$
$$= \frac{\log b_n}{2\pi\sqrt{\det\Gamma}} \left(1 + O\Big(\frac{1}{(\log n)^{1/2}}\Big)\right).$$

We then have that

$$\frac{n}{\mathcal{H}(n)^2}(\mathcal{H}(n) - \mathcal{H}([n/b_n])) \tag{6.21}$$
$$= 2\pi\sqrt{\det\Gamma}\frac{n\log b_n}{(\log n)^2} \left(1 + O\Big(\frac{1}{(\log n)^{1/2}}\Big)\right)$$
$$= 2\pi\sqrt{\det\Gamma}\frac{n\log b_n}{(\log n)^2}(1 + a_n),$$

where we use the last equality to define a_n. Let

$$1 + \widehat{a}_n = (1 + a_n)^{-1} = 1 + O\Big(\frac{1}{(\log n)^{1/2}}\Big). \tag{6.22}$$

Then if we set

$$\widehat{b}_n =: b_n^{1+\widehat{a}_n} = b_n^{(1+a_n)^{-1}} \tag{6.23}$$

we see from (6.21) that

$$\frac{n}{\mathcal{H}(n)^2}(\mathcal{H}(n) - \mathcal{H}([n/\,\widehat{b}_n])) = 2\pi\sqrt{\det\Gamma}\frac{n\log b_n}{(\log n)^2}. \tag{6.24}$$

Also, $\log \widehat{b}_n = (1 + \widehat{a}_n)\log b_n = o((\log n)^{1/2})$, so that Theorem 1.1 applies to $\widehat{b}_n$, and indeed to $\widehat{b}_n^\theta$ for any $\theta > 0$.

Note that

$$\widehat{b}_n^\theta \;=\; b_n^{\theta\left(1+O\left(\frac{1}{(\log n)^{1/2}}\right)\right)} \tag{6.25}$$
$$= b_n^\theta \exp\left(O\Big(\frac{\log b_n}{(\log n)^{1/2}}\Big)\right)$$
$$= b_n^\theta(1 + o(1_n))$$

by our assumption that $\log b_n = o((\log n)^{1/2})$. Hence by (6.24) and (6.25)

$$\widehat{b}_n^{-\theta} \log \mathbb{P}\Big\{\overline{R}_n \geq \frac{\theta n}{\mathcal{H}(n)^2}(\mathcal{H}(n) - \mathcal{H}([n/\widehat{b}_n]))\Big\} \tag{6.26}$$
$$= (1+o(1_n)) b_n^{-\theta} \log \mathbb{P}\Big\{\overline{R}_n \geq 2\pi\theta\sqrt{\det\Gamma}\frac{n\log b_n}{(\log n)^2}\Big\}.$$

Together with Proposition 5.2, Theorem 1.1 applied to $\widehat{b}_n^\theta$ proves the corollary in the strongly aperiodic case. The modifications to handle the case where S_n is not strongly aperiodic are very similar to those in Section 2 of [27]. □

7 Moderate deviations for $\mathbb{E}R_n - R_n$

To avoid difficulties connected with subdividing time intervals, it is more convenient to look at the continuous time analogue of S_n. We let $T_1, T_2, \dots$ be i.i.d. exponential random variables with parameter 1 that are independent of the sequence S_n. Define $Z_t = S_n$ if $\sum_{i=1}^n T_i \leq t < \sum_{i=1}^{n+1} T_i$. Z_t is a Lévy process that waits an exponential length of time, then jumps according to X_1, and then repeats the procedure. Define $N_t = n$ if $\sum_{i=1}^n T_i \leq t < \sum_{i=1}^{n+1} T_i$. Note that N_t is a Poisson process with $\mathbb{E}N_t = t$ and that $Z_t = S_{N_t}$. We write $|Z[a,b]|$ for the cardinality of $\{Z_s : s \in [a,b]\}$.

Theorems 5.3 and 5.4 have the following analogues for continuous time processes. We omit the proofs, which are almost identical to the proofs given for the discrete time random walks.

Lemma 7.1 *Let $Z_1(t), \cdots, Z_p(t)$ be independent copies of $Z(t)$. There is $C > 0$ such that*

$$\sup_{y_1,\cdots,y_p} \mathbb{E}^{(y_1,\cdots,y_p)}\Big|Z_1[0,t]\cap\cdots\cap Z_p[0,t]\Big|^m \leq C^m (m!)^{p-1}\Big(\frac{t}{(\log t)^p}\Big)^m. \tag{7.1}$$

Consequently, there is $\theta > 0$ such that

$$\sup_t \sup_{y_1,\cdots,y_p} \mathbb{E}^{(y_1,\cdots,y_p)} \exp\left\{\theta\Big(\frac{(\log t)^p}{t}\Big|Z_1[0,t]\cap\cdots\cap Z_p[0,t]\Big|\Big)^{(p-1)^{-1}}\right\} < \infty. \tag{7.2}$$

Lemma 7.2 *There is $\theta > 0$ such that*

$$\sup_t \mathbb{E}\exp\Big\{\theta\frac{(\log t)^2}{t}\big|\mathbb{E}|Z[0,t]| - |Z[0,t]|\big|\Big\} < \infty. \tag{7.3}$$

Consequently

$$\limsup_{t\to\infty}\frac{1}{b_t}\log\mathbb{P}\Big\{\Big|\mathbb{E}\big|Z[0,t]\big| - \big|Z[0,t]\big|\Big| \ge \lambda\frac{tb_t}{(\log t)^2}\Big\} \le -\theta\lambda. \tag{7.4}$$

We will prove Theorem 1.5 by first proving the following analogue for Z_t.

Theorem 7.3 *For any $\lambda > 0$ and for any b_t satisfying $b_t \to \infty$ and $b_t = o\Big((\log t)^{1/5}\Big)$ as $t \to \infty$, we have*

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{b_t}\log\mathbb{P}&\left\{\left|\mathbb{E}\big|Z[0,t]\big| - \big|Z[0,t]\big|\right| \ge \lambda\frac{tb_t}{(\log t)^2}\right\}\\ &= -(2\pi)^{-2}\det(\Gamma)^{-1/2}\kappa(2,2)^{-4}\lambda.\end{aligned} \tag{7.5}$$

The next proposition shows that Theorem 1.5 follows from Theorem 7.3 and Theorem 1.1.

Proposition 7.4 *For any $\varepsilon > 0$,*

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\Big|\overline{|Z[0,n]|} - \overline{|S[0,n]|}\Big| \ge \varepsilon\frac{nb_n}{(\log n)^2}\Big\} = -\infty. \tag{7.6}$$

Remark 7.5 Our proof actually gives a stronger result, but this is all we need.

Proof. Observe that if $n > m$, then

$$\Big|\mathbb{E}|S[0,n]| - \mathbb{E}|S[0,m]|\Big| \le \mathbb{E}|S[m,n]| = \mathbb{E}|S[0,n-m]| \le n-m. \tag{7.7}$$

Consequently,

$$\begin{aligned}\Big|\mathbb{E}|Z[0,n]| - \mathbb{E}|S[0,n]|\Big| &= \Big|\mathbb{E}|S[0,N_n]| - \mathbb{E}|S[0,n]|\Big|\\ &\le \mathbb{E}|N_n - n| \le C\sqrt{n}.\end{aligned} \tag{7.8}$$

Hence, it suffices to show that for any $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ \Big| |Z[0,n]| - |S[0,n]| \Big| \ge \varepsilon \frac{\sqrt{n} b_n^{3/2}}{\log n} \Big\} = -\infty. \tag{7.9}$$

Let $M > 0$ be fixed. On the event $\{|N_n - n| \le M\sqrt{nb_n}\}$

$$\begin{aligned} \Big| |Z[0,n]| - |S[0,n]| \Big| &\le \big| S\big[N_n \wedge n, N_n \vee n\big] \big| \\ &\overset{d}{=} \big| S\big[0,\ N_n \vee n - N_n \wedge n\big] \big| \le \big| S\big[0,\ 2M\sqrt{nb_n}\big] \big|. \end{aligned} \tag{7.10}$$

So we have

$$\begin{aligned} &\mathbb{P}\Big\{ \Big| |Z[0,n]| - |S[0,n]| \Big| \ge \varepsilon \frac{\sqrt{n} b_n^{3/2}}{\log n} \Big\} \\ &\le \mathbb{P}\Big\{ \big| S\big[0,\ 2M\sqrt{nb_n}\big] \big| \ge \varepsilon \frac{\sqrt{n} b_n^{3/2}}{\log n} \Big\} + \mathbb{P}\big\{ |N_n - n| \ge M\sqrt{nb_n} \big\}. \end{aligned} \tag{7.11}$$

It follows from (5.30) that

$$\sup_n \mathbb{E} \exp\Big\{ \theta \frac{\log n}{\sqrt{nb_n}} S\big[0,\ 2M\sqrt{nb_n}\big] \Big\} < \infty, \qquad \theta > 0. \tag{7.12}$$

By the Chebyshev inequality one can see that

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ \big| S\big[0,\ 2M\sqrt{nb_n}\big] \big| \ge \varepsilon \frac{\sqrt{n} b_n^{3/2}}{\log n} \Big\} = -\infty. \tag{7.13}$$

We recall the classical moderate deviation principle ([11], Theorem 3.7.1). Let ξ_k be a sequence of i.i.d. such that $\mathbb{E}\xi_1 = 0$ and $\mathbb{E}e^{\lambda|\xi_1|} < \infty$ for some $\lambda > 0$. Then the partial sum $T_n = \xi_1 + \cdots + \xi_n$ obeys the following moderate deviation principle. For any closed set $F \subset \mathbb{R}$

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\{T_n/\sqrt{nb_n} \in F\} \le -\inf_{x\in F} \frac{x^2}{2\sigma^2};$$

for any open set $G \subset \mathbb{R}$

$$\liminf_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\{T_n/\sqrt{nb_n} \in G\} \ge -\inf_{x\in G} \frac{x^2}{2\sigma^2}$$

where $\sigma^2 = \mathbb{E}\xi_1^2$.

Applying that here,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\big\{|N_n - n| \geq M\sqrt{nb_n}\big\} = -\frac{M^2}{2}. \tag{7.14}$$

Thus,

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\left\{\Big||Z[0,n]| - |S[0,n]|\Big| \geq \varepsilon \frac{\sqrt{n} b_n^{3/2}}{\log n}\right\} \leq -\frac{M^2}{2}. \tag{7.15}$$

Letting $M \to \infty$ proves the proposition. □

Thus we we need to prove Theorem 7.3. We recall the Gärtner-Ellis theorem ([11, Theorem 2.3.6]).

Theorem 7.6 (Gärtner-Ellis) *Let Y_n be a sequence of random variables and b_n be a positive sequence such that $b_n \longrightarrow \infty$ and such that the limit*

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\Big\{\theta b_n Y_n\Big\} = \Lambda(\theta)$$

exists for all $\theta \in \mathbb{R}$. Assume that $\Lambda(\theta)$ is differentiable and that $\Lambda'(\theta) \longrightarrow \infty$ as $\theta \to \pm\infty$. Then for any closed set $F \subset \mathbb{R}$

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\{Y_n \in F\} \leq -\inf_{x\in F} \Lambda^*(\lambda)$$

and for any open set $G \subset \mathbb{R}$

$$\liminf_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\{Y_n \in G\} \geq -\inf_{x\in G} \Lambda^*(\lambda)$$

where

$$\Lambda^*(\lambda) = \sup_{\theta\in\mathbb{R}} \{\theta\lambda - \Lambda(\theta)\}.$$

Thus to prove Theorem 7.3 it suffices to prove

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{b_t} \log \mathbb{E} \exp\left\{\theta \sqrt{\frac{b_t}{t}} (\log t) \Big|\mathbb{E}\big|Z[0,t]\big| - \big|Z[0,t]\big|\Big|^{1/2}\right\} \\ &= (\theta\pi)^2 \sqrt{\det(\Gamma)}\kappa(2,2)^4. \end{aligned} \tag{7.16}$$

Let $h(x)$ be a smooth symmetric probability density on $\mathbb{R}^2$ with compact support and write $h_\varepsilon(x) = \varepsilon^{-2}h(\varepsilon^{-1}x)$. We have

$$\Lambda_\varepsilon(t) \equiv \sum_{x\in\mathbb{Z}^2} h_\varepsilon\Big(\frac{x}{\sqrt{t}}\Big) \sim t, \qquad t\to\infty. \tag{7.17}$$

The following lemma describing exponential asymptotics for the smoothed range will be proved in Section 8.

Lemma 7.7 *Let*

$$A_t(\varepsilon) \equiv \Lambda_\varepsilon\Big(\frac{t}{b_t}\Big)^{-2} \sum_{x\in\mathbb{Z}^2}\Big[\sum_{y\in Z[0,t]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y)\Big)\Big]^2. \tag{7.18}$$

For any $\theta > 0$,

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{b_t}\log\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_t}{t}}(\log t)\,|A_t(\varepsilon)|^{1/2}\Big\}\\ = \sup_{g\in\mathcal{F}}\Big\{2\pi\theta\sqrt{\det(\Gamma)}\Big(\int_{\mathbb{R}^2}|(g^2*h_\varepsilon)(x)|^2dx\Big)^{1/2}\\ -\frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x),\Gamma\nabla g(x)|^2dx\Big\}.\end{aligned} \tag{7.19}$$

where

$$\mathcal{F} = \{g\in W^{1,2}(\mathbb{R}^2);\ \ ||g||_2 = 1\}.$$

Furthermore, for any $N = 0,1,\dots$ *and any* $\epsilon > 0$,

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{b_t}\log\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_t}{t}}(\log t)\\ \times\Big(\Lambda_\varepsilon\Big(\frac{t}{b_t}\Big)^{-2}\sum_{x\in\mathbb{Z}^2}\Big[\sum_{y\in Z[0,2^{-N}t]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y)\Big)\Big]^2\Big)^{1/2}\Big\}\\ \le 2^{-N+2}\pi^2\theta^2\sqrt{\det(\Gamma)}\kappa(2,2)^4.\end{aligned} \tag{7.20}$$

The following lemma on exponential approximation will be proved in Section 9. In this lemma Z' denotes an independent copy of Z.

Lemma 7.8 *Let*

$$B_t^{(j)}(\varepsilon) \equiv \Lambda_\varepsilon \Big(\frac{t}{b_t}\Big)^{-2}$$

$$\times \sum_{x\in\mathbb{Z}^2} \Big[\sum_{y\in Z[0,2^{-j}t]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y)\Big)\Big]\Big[\sum_{y'\in Z'[0,2^{-j}t]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y')\Big)\Big]. \tag{7.21}$$

Then for any $\theta > 0$ and any $j = 0, 1, \dots,$

$$\limsup_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{b_t}$$

$$\log\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_t}{t}}(\log t)\Big|\,|Z[0,2^{-j}t]\cap Z'[0,2^{-j}t]| - B_t^{(j)}(\varepsilon)\Big|^{1/2}\Big\} = 0. \tag{7.22}$$

These lemmas will be the key to proving Theorem 7.3. Before proving this theorem, we present a simple lemma which will be used several times in the proof of Theorem 7.3.

Lemma 7.9 *Let $l \geq 2$ be a fixed integer and let $\{\xi_1(\rho);\ \rho > 0\}, \cdots, \{\xi_l(\rho);\ \rho > 0\}$ be l independent non-negative stochastic processes.*

(a) If there is a constant $C_1 > 0$ such that for any $1 \leq j \leq l$,

$$\limsup_{\rho\to 0^+}\rho\log\mathbb{P}\big\{\xi_j(\rho)\geq\lambda\big\} \leq -C_1\lambda, \qquad \lambda > 0, \tag{7.23}$$

then

$$\limsup_{\rho\to 0^+}\rho\log\mathbb{P}\Big\{\xi_1(\rho)+\cdots+\xi_l(\rho)\geq\lambda\Big\} \leq -C_1\lambda, \qquad \lambda > 0. \tag{7.24}$$

(b) If there is a constant $C_2 > 0$ such that for any $1 \leq j \leq l$,

$$\limsup_{\rho\to 0^+}\rho\log\mathbb{E}\exp\Big\{\rho^{-1}\theta\sqrt{\xi_j(\rho)}\Big\} \leq C_2\theta^2, \qquad \theta > 0, \tag{7.25}$$

then

$$\limsup_{\rho\to 0^+}\rho\log\mathbb{E}\exp\Big\{\rho^{-1}\theta\sqrt{\xi_1(\rho)+\cdots+\xi_l(\rho)}\Big\} \leq C_2\theta^2, \qquad \theta > 0. \tag{7.26}$$

Proof. . Clearly, part (a) needs only to be proved in the case $l = 2$. Given $0 < \delta < \lambda$, let $0 = a_0 < a_1 < \cdots < a_N = \lambda$ be a partition of $[0, \lambda]$ such that $a_k - a_{k-1} < \delta$. Then

$$\begin{aligned} \mathbb{P}\big\{\xi_1(\rho) + \xi_2(\rho) \geq \lambda\big\} &\leq \sum_{k=1}^{N} \mathbb{P}\big\{\xi_1(\rho) \in [a_{k-1}, a_k]\big\}\mathbb{P}\big\{\xi_2(\rho) \geq \lambda - a_k\big\} \\ &\leq \sum_{k=1}^{N} \mathbb{P}\big\{\xi_1(\rho) \geq a_{k-1}\big\}\mathbb{P}\big\{\xi_2(\rho) \geq \lambda - a_k\big\}. \end{aligned} \tag{7.27}$$

Hence

$$\begin{aligned} &\limsup_{\rho\to 0^+} \rho \log \mathbb{P}\big\{\xi_1(\rho) + \xi_2(\rho) \geq \lambda\big\} \\ &\leq \max_{1\leq k\leq N} \Big\{ - C_1 a_{k-1} - C_1(\lambda - a_k)\Big\} \leq -C_1(\lambda - \delta). \end{aligned} \tag{7.28}$$

Letting $\delta \to 0^+$ proves part (a).

We now prove part (b). By Chebyshev's inequality, for any $\lambda > 0$

$$\limsup_{\rho\to 0^+} \rho \log \mathbb{P}\big\{\xi_j(\rho) \geq \lambda\big\} \leq -\sup_{\theta>0}\{\theta\sqrt{\lambda} - C_2\theta^2\} = -\frac{\lambda}{4C_2}. \tag{7.29}$$

By part (a)

$$\limsup_{\rho\to 0^+} \rho \log \mathbb{P}\Big\{\xi_1(\rho) + \cdots + \xi_l(\rho) \geq \lambda\Big\} \leq -\frac{\lambda}{4C_2}, \qquad \lambda > 0. \tag{7.30}$$

In addition, by the triangle inequality and by independence,

$$\mathbb{E} \exp\Big\{\rho^{-1}\theta\sqrt{\xi_1(\rho) + \cdots + \xi_l(\rho)}\Big\} \leq \prod_{j=1}^{l} \mathbb{E} \exp\Big\{\rho^{-1}\theta\sqrt{\xi_j(\rho)}\Big\}. \tag{7.31}$$

So by assumption, for any $\theta > 0$,

$$\limsup_{\rho\to 0^+} \rho \log \mathbb{E} \exp\Big\{\rho^{-1}\theta\sqrt{\xi_1(\rho) + \cdots + \xi_l(\rho)}\Big\} < \infty. \tag{7.32}$$

We will need the following Lemma. See [11, Section 4.3] for a proof.

Lemma 7.10 (Varadhan's integral lemma) *Let Y_n be a sequence of random variables and let b_n be a positive sequence such that $b_n \longrightarrow \infty$. Let I: $\mathbb{R} \longrightarrow [0,\infty]$ be a good rate function (which means that $I(x) \longrightarrow \infty$ as $|x| \to \infty$, and that $I(\cdot)$ is lower semi-continuous).*

(1). Assume that Y_n satisfies the upper bound of the large deviation principle: For any closed set $F \subset \mathbb{R}$

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\{Y_n \in F\} \le -\inf_{x\in F} I(x)$$

and also the uniform exponential integrability

$$\lim_{M\to\infty} \limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}\Big[\exp\exp\Big\{b_n\varphi(Y_n)\Big\}1_{\{\varphi(Y_n)\ge M\}}\Big] = -\infty.$$

Then for any upper semi-continuous function φ on $\mathbb{R}$,

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}\exp\Big\{\theta b_n\varphi(Y_n)\Big\} \le \sup_{x\in\mathbb{R}}\{\varphi(x) - I(x)\}.$$

(2) Assume that Y_n satisfies the lower bound of the large deviation principle: For any open set $G \subset \mathbb{R}$

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\{Y_n \in G\} \ge -\inf_{x\in G} I(x).$$

Then for any lower semi-continuous function φ on $\mathbb{R}$,

$$\liminf_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}\exp\Big\{\theta b_n\varphi(Y_n)\Big\} \ge \sup_{x\in\mathbb{R}}\{\varphi(x) - I(x)\}.$$

(3) Assume that Y_n satisfies the upper and lower bounds of the large deviation principle, uniform exponential integrability and that φ is continuous. Then

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}\exp\Big\{\theta b_n\varphi(Y_n)\Big\} = \sup_{x\in\mathbb{R}}\{\varphi(x) - I(x)\}.$$

We apply part (1) of Varadhan's integral lemma to our setting by identifying b_n with ρ^{-1} and Y_n with $\xi_1(\rho) + \cdots + \xi_l(\rho)$. By Hölder's inequality, (7.32) implies uniform exponential integrability. All we need to check now is that (7.30) leads to the upper bound of the large deviation principle in the form given in Varadhan's lemma. Let $0 < a < b < \infty$ be fixed and write $Y(\rho) = \xi_1(\rho) + \cdots + \xi_l(\rho)$, $I(\lambda) = \frac{\lambda}{4C_2}$. Note that

$$\mathbb{P}\{Y(\rho) \geq a\} \geq \mathbb{P}\{Y(\rho) \in (a,b)\}.$$

By (7.30),

$$\limsup_{\rho\to 0^+} \rho \log \mathbb{P}\{Y(\rho) \in (a,b)\} \leq -I(a) = -\inf_{\lambda\in(a,b)} I(\lambda).$$

Since F is compact, for any given $\delta > 0$, F can be covered by finitely many open intervals with diameters less than δ. Let F^δ be the union of these open intervals. Then we have

$$\limsup_{\rho\to 0^+} \rho \log \mathbb{P}\{Y(\rho) \in F\} \leq -\inf_{\lambda\in F^\delta} I(\lambda).$$

Letting $\delta \to 0^+$ on the right hand side proves that

$$\limsup_{\rho\to 0^+} \rho \log \mathbb{P}\{Y(\rho) \in F\} \leq -\inf_{\lambda\in F} I(\lambda).$$

To extend the above upper bound to general F, notice that for any $b > 0$

$$\mathbb{P}\{Y(\rho) \in F\} \leq \mathbb{P}\{Y(\rho) \in F_b\} + \mathbb{P}\{Y(\rho) \geq b\}$$

where $F_b = F \cap [0,b]$. Then

$$\begin{aligned}\limsup_{\rho\to 0^+} \rho \log \mathbb{P}\{Y(\rho) \in F\} &\leq \max\Big\{-\inf_{\lambda\in F_b} I(\lambda), \quad -I(b)\Big\}\\ &\leq \max\Big\{-\inf_{\lambda\in F} I(\lambda), \quad -I(b)\Big\}.\end{aligned}$$

Letting $b \to \infty$ on the right hand side gives the desired upper bound:

$$\begin{aligned}&\limsup_{\rho\to 0^+} \rho \log \mathbb{E} \exp\Big\{\rho^{-1}\theta\sqrt{\xi_1(\rho) + \cdots + \xi_l(\rho)}\Big\}\\ &\leq \sup_{\lambda>0}\Big\{\theta\sqrt{\lambda} - \frac{\lambda}{4C_2}\Big\} = C_2\theta^2.\end{aligned} \tag{7.33}$$

$\square$

Proof of Theorem 7.3: We begin with the decomposition

$$\begin{aligned}\big|Z[0,t]\big| &= \sum_{k=1}^{2^N}\Big|Z\Big[\frac{k-1}{2^N}t,\ \frac{k}{2^N}t\Big]\Big|\\ &\quad-\sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}}\Big|Z\Big[\frac{2k-2}{2^j}t,\ \frac{2k-1}{2^j}t\Big]\cap Z\Big[\frac{2k-1}{2^j}t,\ \frac{2k}{2^j}t\Big]\Big| \qquad (7.34)\\ &=: I_t - J_t.\end{aligned}$$

We first establish the upper bound. Let $\varepsilon > 0$ be fixed. Since

$$E\big|Z[0,t]\big| - \big|Z[0,t]\big| = (\mathbb{E}I_t - I_t) + J_t - \mathbb{E}J_t \le (\mathbb{E}I_t - I_t) + J_t, \qquad (7.35)$$

it follows that

$$\begin{aligned}\mathbb{P}\Big\{\Big|\mathbb{E}\big|Z[0,t]\big| - \big|Z[0,t]\big|\Big| \ge \lambda t b_t/(\log t)^2\Big\} \qquad (7.36)\\ \le \mathbb{P}\Big\{\big|\mathbb{E}I_t - I_t\big| \ge \varepsilon t b_t/(\log t)^2\Big\} + \mathbb{P}\Big\{J_t \ge (\lambda-\varepsilon) t b_t/(\log t)^2\Big\}.\end{aligned}$$

Notice that

$$\big|\mathbb{E}I_t - I_t\big| \le \sum_{k=1}^{2^N}\Big|\mathbb{E}\Big|Z\Big[\frac{k-1}{2^N}t,\ \frac{k}{2^N}t\Big]\Big| - \Big|Z\Big[\frac{k-1}{2^N}t,\ \frac{k}{2^N}t\Big]\Big|\Big|. \qquad (7.37)$$

Replacing t by $2^{-N}t$, λ by $2^N\lambda$ and b_t by $\widetilde{b}_t =: b_{2^N t}$ in (7.4) we obtain

$$\limsup_{t\to\infty}\frac{1}{b_t}\log\mathbb{P}\Big\{\Big|\mathbb{E}\big|Z[0,2^{-N}t]\big| - \big|Z[0,2^{-N}t]\big|\Big| \ge \lambda\frac{tb_t}{(\log t)^2}\Big\} \le -2^N C\lambda. \qquad (7.38)$$

Hence by Lemma 7.9,

$$\limsup_{t\to\infty}\frac{1}{b_t}\log\mathbb{P}\Big\{\big|\mathbb{E}I_t - I_t\big| \ge \frac{\varepsilon t b_t}{(\log t)^2}\Big\} \le -\varepsilon C 2^N. \qquad (7.39)$$

By the triangle inequality,

$$\mathbb{P}\Big\{J_t \ge \frac{(\lambda-\varepsilon)tb_t}{(\log t)^2}\Big\} \le \sum_{j=1}^{N}\mathbb{P}\Big\{\sum_{k=1}^{2^{j-1}}\xi_{j,k} \ge 2^{-j}\frac{(\lambda-\varepsilon)tb_t}{(\log t)^2}\Big\} \qquad (7.40)$$

where for each $1 \le j \le N$,
(7.41)

$$\xi_{j,k}(t) = \Big| Z\Big[\frac{2k-2}{2^j}t,\ \frac{2k-1}{2^j}t\Big] \cap Z\Big[\frac{2k-1}{2^j}t,\ \frac{2k}{2^j}t\Big]\Big|, \qquad k = 1, \cdots, 2^{j-1},$$

forms an i.i.d. sequence with the same distribution as

$$|Z[0, 2^{-j}t] \cap Z'[0, 2^{-j}t]|. \tag{7.42}$$

By [8, Theorem 1]

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ \#\{S^{(1)}[1,n] \cap S^{(2)}[1,n]\} \ge \lambda \frac{n}{(\log n)^2} b_n \Big\}$$
$$= -(2\pi)^{-2} \det(\Gamma)^{-\frac{1}{2}} \kappa(2,2)^{-4}\lambda$$

where $\kappa(2,2)$ is the best constant of the following Gagliardo-Nirenberg inequality

$$||f||_4 \le C||\nabla f||_2^{\frac{1}{2}} ||f||_2^{1/2}, \qquad f \in W^{1,2}(\mathbb{R}^2).$$

The same result holds if the random walks are replaced by the lattice valued Lévy processes so that (with $2^{-j}t$ instead of t), for any $\lambda > 0$,

$$\lim_{t\to\infty} \frac{1}{b_t} \log \mathbb{P}\Big\{ |Z[0, 2^{-j}t] \cap Z'[0, 2^{-j}t]| \ge \frac{\lambda t b_t}{(\log t)^2} \Big\}$$
$$= -2^j (2\pi)^{-2} \det(\Gamma)^{-1/2} \kappa(2,2)^{-4}\lambda. \tag{7.43}$$

Therefore, by Lemma 7.9,

$$\lim_{t\to\infty} \frac{1}{b_t} \log \mathbb{P}\Big\{ \sum_{k=1}^{2^{j-1}} \xi_{j,k} \ge \frac{\lambda t b_t}{(\log t)^2} \Big\} = -2^j (2\pi)^{-2} \det(\Gamma)^{-1/2} \kappa(2,2)^{-4}\lambda. \tag{7.44}$$

In particular,

$$\lim_{t\to\infty} \frac{1}{b_t} \log \mathbb{P}\Big\{ \sum_{k=1}^{2^{j-1}} \xi_{j,k} \ge 2^{-j} \frac{(\lambda-\varepsilon) t b_t}{(\log t)^2} \Big\}$$
$$= -(2\pi)^{-2} \det(\Gamma)^{-1/2} \kappa(2,2)^{-4} (\lambda - \varepsilon) \tag{7.45}$$

and therefore by (7.40)

$$\lim_{t\to\infty} \frac{1}{b_t} \log \mathbb{P}\Big\{ J_t \ge \frac{(\lambda-\varepsilon) t b_t}{(\log t)^2} \Big\} = -(2\pi)^{-2} \det(\Gamma)^{-1/2} \kappa(2,2)^{-4} (\lambda-\varepsilon). \tag{7.46}$$

Combining (7.36), (7.39) and (7.46) and letting $\varepsilon \to 0$ we obtain

$$\limsup_{t\to\infty} \frac{1}{b_t} \log \mathbb{P}\Big\{ \Big| \mathbb{E}\big|Z[0,t]\big| - \big|Z[0,t]\big| \Big| \ge \frac{\lambda t b_t}{(\log t)^2} \Big\}$$

$$\le -(2\pi)^{-2} \det(\Gamma)^{-1/2} \kappa(2,2)^{-4} \lambda. \tag{7.47}$$

Using Lemma 7.10

$$\limsup_{t\to\infty} \tfrac{1}{b_t} \log \mathbb{E} \exp \left\{ \theta \sqrt{\tfrac{b_t}{t}} (\log t) \Big| \mathbb{E}\big|Z[0,t]\big| - \big|Z[0,t]\big| \Big|^{1/2} \right\} \tag{7.48}$$

$$\le \sup_{\lambda>0} \left\{ \theta\lambda^{1/2} - (2\pi)^{-2} \det(\Gamma)^{-1/2} \kappa(2,2)^{-4} \lambda \right\}$$

$$= (\theta\pi)^2 \sqrt{\det(\Gamma)} \kappa(2,2)^4.$$

(The uniform exponential integrability is provided by Lemma 7.2.)

We now prove the lower bound. Using induction on N, one can see that

$$A_t(\varepsilon) =: \Lambda_\varepsilon \Big(\frac{t}{b_t}\Big)^{-2} \sum_{x\in\mathbb{Z}^2} \Big[\sum_{y\in Z[0,t]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y)\Big)\Big]^2$$

$$\le \Lambda_\varepsilon \Big(\frac{t}{b_t}\Big)^{-2} \sum_{k=1}^{2^N} \sum_{x\in\mathbb{Z}^2} \Big[\sum_{y\in Z\left[\frac{k-1}{2^N}t,\ \frac{k}{2^N}t\right]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y)\Big)\Big]^2$$

$$+2\Lambda_\varepsilon \Big(\frac{t}{b_t}\Big)^{-2} \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} \sum_{x\in\mathbb{Z}^2} \Big[\sum_{y\in Z\left[\frac{2k-2}{2^j}t,\ \frac{2k-1}{2^j}t\right]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y)\Big)\Big]$$

$$\times \Big[\sum_{y'\in Z\left[\frac{2k-1}{2^j}t,\ \frac{2k}{2^j}t\right]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y')\Big)\Big]$$

$$=: I_t(\varepsilon) + 2J_t(\varepsilon). \tag{7.49}$$

Therefore, with I_t, J_t given by (7.34)

$$\mathbb{E}\big|Z[0,t]\big| - \big|Z[0,t]\big| = (\mathbb{E}I_t - I_t) + J_t - \mathbb{E}J_t \tag{7.50}$$

$$\ge (\mathbb{E}I_t - I_t) + J_t(\varepsilon) - |J_t - J_t(\varepsilon)| - \mathbb{E}J_t$$

$$\ge (\mathbb{E}I_t - I_t) - \tfrac{1}{2}I_t(\varepsilon) - |J_t - J_t(\varepsilon)| - \mathbb{E}J_t + \tfrac{1}{2}A_t(\varepsilon).$$

We will see that the dominant contribution to the lower bound comes from $A_t(\varepsilon)$. By the last display we see that

$$\frac{1}{2}A_t(\varepsilon) \le \big|\mathbb{E}\big|Z[0,t]\big| - \big|Z[0,t]\big|\big| + |\mathbb{E}I_t - I_t| + \frac{1}{2}I_t(\varepsilon) + |J_t - J_t(\varepsilon)| + \mathbb{E}J_t. \tag{7.51}$$

and consequently

$$\begin{aligned}\left|\frac{1}{2}A_t(\varepsilon)\right|^{1/2} &\le \left|\mathbb{E}\big|Z[0,t]\big| - \big|Z[0,t]\big|\right|^{1/2} + |\mathbb{E}I_t - I_t|^{1/2} \\ &\quad + \left|\frac{1}{2}I_t(\varepsilon)\right|^{1/2} + |J_t - J_t(\varepsilon)|^{1/2} + |\mathbb{E}J_t|^{1/2}.\end{aligned} \tag{7.52}$$

Notice that it follows from (7.1) that

$$\mathbb{E}J_t \le C_N \frac{t}{(\log t)^2}. \tag{7.53}$$

If $\bar{p}$ is such that $p^{-1} + \bar{p}^{-1} = 1$, then by the generalized Hölder inequality with $f = \theta\sqrt{\frac{b_t}{t}}\log t$ we have

$$\begin{aligned}&\left\|\exp\frac{f}{p}\big|\frac{1}{2}A_t(\varepsilon)\big|^{1/2}\right\|_1 \\ &\le e^{C_N\sqrt{b_t}}\left\|\exp\frac{f}{p}\big|\mathbb{E}\big|Z[0,t]\big| - \big|Z[0,t]\big|\big|^{1/2}\right\|_p \cdot \left\|\exp\frac{f}{p}|\mathbb{E}I_t - I_t|^{1/2}\right\|_{3\bar{p}} \\ &\qquad \cdot\left\|\exp\frac{f}{p}\big|\frac{1}{2}I_t(\varepsilon)\big|^{1/2}\right\|_{3\bar{p}} \cdot \left\|\exp\frac{f}{p}|J_t - J_t(\varepsilon)|^{1/2}\right\|_{3\bar{p}}\end{aligned} \tag{7.54}$$

Taking the p-th power and noting that $\bar{p}/p = 1/(p-1)$, this can be rewritten as

$$\begin{aligned}
(7.55)\qquad & \mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_t}{t}}(\log t)\Big|\mathbb{E}\big|Z[0,t]\big| - \big|Z[0,t]\big|\Big|^{1/2}\Big\} \\
&\geq e^{-C_N\sqrt{b_t}}\left[\mathbb{E}\exp\left\{\frac{3\theta}{p-1}\sqrt{\frac{b_t}{t}}(\log t)|\mathbb{E}I_t - I_t|^{1/2}\right\}\right]^{-\frac{p-1}{3}} \\
&\quad\times\left[\mathbb{E}\exp\left\{\frac{3\theta}{p-1}\sqrt{\frac{b_t}{t}}(\log t)I_t(\varepsilon)^{1/2}\right\}\right]^{-\frac{p-1}{3}} \\
&\quad\times\left[\mathbb{E}\exp\left\{\frac{3\theta}{p-1}\sqrt{\frac{b_t}{t}}(\log t)|J_t - J_t(\varepsilon)|^{1/2}\right\}\right]^{-\frac{p-1}{3}} \\
&\quad\times\left[\mathbb{E}\exp\left\{\frac{\theta}{2p}\sqrt{\frac{b_t}{t}}(\log t)|A_t(\varepsilon)|^{1/2}\right\}\right]^{p}.
\end{aligned}$$

By Lemma 7.7

$$\begin{aligned}
(7.56)\qquad & \lim_{t\to\infty}\frac{1}{b_t}\log\mathbb{E}\exp\left\{\frac{\theta}{2p}\sqrt{\frac{b_t}{t}}(\log t)|A_t(\varepsilon)|^{1/2}\right\} \\
&= \sup_{g\in\mathcal{F}}\left\{\frac{\pi\theta}{p}\sqrt{\det(\Gamma)}\left(\int_{\mathbb{R}^2}|(g^2 * h_\varepsilon)(x)|^2dx\right)^{1/2}\right. \\
&\quad\left. - \frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x), \Gamma\nabla g(x)\rangle dx\right\}.
\end{aligned}$$

This will give the main contribution to (7.55). We now bound the other factors in (7.55).

Using Lemma 7.9 together with (7.48) (with t replaced by $2^{-N}t$, θ by $2^{-N/2}\theta$, and b_t by $\widetilde{b}_t =: b_{2^N t}$) we can prove that for any $\theta > 0$,

$$(7.57)\qquad \limsup_{t\to\infty}\frac{1}{b_t}\log\mathbb{E}\exp\left\{\theta\sqrt{\frac{b_t}{t}}(\log t)|\mathbb{E}I_t - I_t|^{1/2}\right\} \leq 2^{-N}C\theta^2.$$

Using (7.20) and Lemma 7.9, we see that

$$(7.58)\qquad \limsup_{t\to\infty}\frac{1}{b_t}\log\mathbb{E}\exp\left\{\theta\sqrt{\frac{b_t}{t}}(\log t)I_t(\varepsilon)^{1/2}\right\} \leq 2^{-N}C\theta^2,$$

where $C > 0$ does not depend on ε. Notice that

$$|J_t - J_t(\varepsilon)| \le \sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}} |K_{j,k}(\varepsilon)|, \tag{7.59}$$

where

$$K_{j,k}(\varepsilon) = \Big| Z\Big[\frac{2k-2}{2^j}t,\ \frac{2k-1}{2^j}t\Big] \cap Z\Big[\frac{2k-1}{2^j}t,\ \frac{2k}{2^j}t\Big]\Big|$$
$$- \Lambda_\varepsilon\Big(\frac{t}{b_t}\Big)^{-2} \sum_{x\in\mathbb{Z}^2}\Big[\sum_{y\in Z\left[\frac{2k-2}{2^j}t,\ \frac{2k-1}{2^j}t\right]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y)\Big)\Big]$$
$$\times \Big[\sum_{y'\in Z\left[\frac{2k-1}{2^j}t,\ \frac{2k}{2^j}t\right]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y')\Big)\Big]. \tag{7.60}$$

For each $1\le j\le N$, $K_{j,1}(\varepsilon),\cdots,K_{j,2^{N-1}}(\varepsilon)$ forms an i.i.d sequence with the same distribution as $B_t^{(j)}(\varepsilon)$. It then follows from Lemma 7.8 and Hölder's inequality that

$$\limsup_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{b_t}\log\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_t}{t}}(\log t)|J_t - J_t(\varepsilon)|^{1/2}\Big\} = 0. \tag{7.61}$$

Hence

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{b_t}\log\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_t}{t}}(\log t)\Big|\mathbb{E}\big|Z[0,t]\big| - \big|Z[0,t]\big|\Big|^{1/2}\Big\}\\
&\quad\ge -2^{-N+1}C\frac{p-1}{3}\Big(\frac{3\theta}{p-1}\Big)^2\\
&\qquad - \frac{p-1}{3}\limsup_{t\to\infty}\frac{1}{b_t}\log\mathbb{E}\exp\Big\{\frac{3\theta}{p-1}\sqrt{\frac{b_t}{t}}\log t|J_t - J_t(\varepsilon)|^{1/2}\Big\}\\
&\qquad + p\sup_{g\in\mathcal{F}}\Big\{\frac{\pi\theta}{p}\sqrt{\det(\Gamma)}\Big(\int_{\mathbb{R}^2}|(g^2 * h_\varepsilon)(x)|^2dx\Big)^{1/2}\\
&\qquad - \frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x),\Gamma\nabla g(x)\rangle dx\Big\}.
\end{aligned}$$

Take limits on the right hand side in the following order: let $\varepsilon\to 0^+$, (using

(7.61)), $N \to \infty$, and then $p \to 1^+$. We obtain

(7.62)

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{b_t}\log\mathbb{E}\exp\left\{\theta\sqrt{\frac{b_t}{t}}(\log t)\Big|\mathbb{E}\big|Z[0,t]\big|-\big|Z[0,t]\big|\Big|^{1/2}\right\}\\
&\quad\geq\sup_{g\in\mathcal{F}}\left\{\pi\theta\sqrt{\det(\Gamma)}\bigg(\int_{\mathbb{R}^2}|g(x)|^4dx\bigg)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x),\Gamma\nabla g(x)\rangle dx\right\}\\
&\quad=(\pi\theta)^2\sqrt{\det(\Gamma)}\sup_{f\in\mathcal{F}}\left\{\bigg(\int_{\mathbb{R}^2}|f(x)|^4dx\bigg)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla f(x)|^2dx\right\}\\
&\quad=(\pi\theta)^2\sqrt{\det(\Gamma)}\kappa(2,2)^4,
\end{aligned}$$

where the second step follows from the substitution $g(x)=\sqrt{|\det(A)|}f(Ax)$ with the 2×2 matrix A satisfying

$$A^{\tau}\Gamma A=(\pi\theta)^2\sqrt{\det(\Gamma)}I_{2\times 2} \tag{7.63}$$

($I_{2\times 2}$ is the 2×2 identity matrix), and where the last step follows from the identity

$$\sup_{f\in\mathcal{F}}\left\{\bigg(\int_{\mathbb{R}^2}|f(x)|^4dx\bigg)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla f(x)|^2dx\right\}=\kappa(2,2)^4 \tag{7.64}$$

which is a special form of Lemma A.2 in [7]. Here we give a proof.

For any $f\in\mathcal{F}$,

$$\begin{aligned}
&\bigg(\int_{\mathbb{R}^2}|f(x)|^4dx\bigg)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla f(x)|^2dx\\
&\leq\kappa(2,2)^2\bigg(\int_{\mathbb{R}^2}|\nabla f(x)|^2dx\bigg)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla f(x)|^2dx\\
&\leq\sup_{\theta>0}\{\kappa(2,2)^2\theta-\frac{1}{2}\theta^2\}=\kappa(2,2)^4
\end{aligned}$$

so that

$$\sup_{f\in\mathcal{F}}\left\{\bigg(\int_{\mathbb{R}^2}|f(x)|^4dx\bigg)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla f(x)|^2dx\right\}\leq\kappa(2,2)^4.$$

On the other hand, for any $C < \kappa(2,2)$, there is a $g(x)$ on $\mathbb{R}^2$ such that

$$||g||_4 > C||\nabla g||_2^{1/2}||g||_2^{1/2}.$$

By homogeneity, we may assume that $||g|_2 = 1$. Given $\lambda > 0$, let $f(x) = \lambda g(\lambda x)$. Then $||f||_2 = 1$, $||\nabla f||_2 = \lambda||\nabla g||_2$ and

$$||f||_4 = \lambda^{1/2}||g||_4 \ge C\Big(\lambda||\nabla g||_2\Big)^{1/2}.$$

Hence,

$$\begin{aligned}&\sup_{f\in\mathcal{F}}\left\{\left(\int_{\mathbb{R}^2}|f(x)|^4dx\right)^{1/2} - \frac{1}{2}\int_{\mathbb{R}^2}|\nabla f(x)|^2dx\right\}\\ &\ge C^2\lambda||\nabla g||_2 - \frac{1}{2}\Big(\lambda||\nabla g||_2\Big)^2.\end{aligned}$$

Since $\lambda > 0$ is arbitrary,

$$\begin{aligned}&\sup_{f\in\mathcal{F}}\left\{\left(\int_{\mathbb{R}^2}|f(x)|^4dx\right)^{1/2} - \frac{1}{2}\int_{\mathbb{R}^2}|\nabla f(x)|^2dx\right\}\\ &\ge \sup_{\theta>0}\{C^2\theta - \frac{1}{2}\theta^2\} = C^4.\end{aligned}$$

This completes the proof of (7.64). □

8 Exponential asymptotics for the smoothed range

In order to prove Lemma 7.7 we first obtain a weak convergence result.

Let $\beta > 0$ and write

$$A_{t,\beta}(\varepsilon) =: \Lambda_\varepsilon(t)^{-2}\sum_{x\in\mathbb{Z}^2}\Big[\sum_{y\in Z[0,\beta t]} h_\varepsilon\Big(\frac{x-y}{\sqrt{t}}\Big)\Big]^2 \tag{8.1}$$

and

$$B_{t,\beta}(\varepsilon) =: \Lambda_\varepsilon(t)^{-2}\sum_{x\in\mathbb{Z}^2}\Big[\sum_{y\in Z[0,\beta t]} h_\varepsilon\Big(\frac{x-y}{\sqrt{t}}\Big)\Big]\Big[\sum_{y'\in Z'[0,\beta t]} h_\varepsilon\Big(\frac{x-y'}{\sqrt{t}}\Big)\Big]. \tag{8.2}$$

Let $W(t), W'(t)$ be independent planar Brownian motions, each with covariance matrix Γ and write

$$\alpha_\varepsilon([0,t]^2) = \int_0^t \int_0^t (h_\varepsilon * h_\varepsilon)\big(W(s) - W'(r)\big) dr\, ds \tag{8.3}$$

and

$$\alpha([0,t]^2) = \lim_{\varepsilon \to 0} \alpha_\varepsilon([0,t]^2). \tag{8.4}$$

Lemma 8.1

$$\begin{aligned}\frac{(\log t)^2}{t} & \Big[\big| Z[0,\beta t] \cap Z'[0,\beta t] \big| - B_{t,\beta}(\varepsilon) \Big] \\ & \xrightarrow{d} (2\pi)^2 \det(\Gamma) \Big[\alpha([0,\beta]^2) - \alpha_\varepsilon([0,\beta]^2) \Big]\end{aligned} \tag{8.5}$$

and

$$\frac{(\log t)^2}{t} A_{t,\beta}(\varepsilon) \xrightarrow{d} (2\pi)^2 \det(\Gamma) \int_{\mathbb{R}^2} \bigg(\int_0^\beta h_\varepsilon\big(W(s) - x\big) ds \bigg)^2 dx. \tag{8.6}$$

as $t \to \infty$.

Proof. To prove (8.5), we consider the following result given on p.697 of [27]: if $Z^{(t)}(s) =: \frac{Z(ts)}{\sqrt{t}}$ then

$$\begin{aligned}&\Big(Z^{(t)}(\cdot), (Z')^{(t)}(\cdot), \frac{(\log t)^2}{t} \big| Z[0,\beta t] \cap Z'[0,\beta t] \big| \Big) \\ &\qquad \xrightarrow{d} \Big(W(\cdot), W'(\cdot), (2\pi)^2 \det(\Gamma) \alpha([0,\beta]^2) \Big)\end{aligned} \tag{8.7}$$

in the Skorokhod topology as $t \to \infty$. Actually, the proof in [27] is for the discrete time random walk, but a similar proof works for Z.

Let $M > 0$ be fixed for a moment. Notice that

$$p_{t,\varepsilon}(x) \equiv \Lambda_\varepsilon(t)^{-1} h_\varepsilon\Big(\frac{x}{\sqrt{t}}\Big), \quad x \in \mathbb{Z}^2, \tag{8.8}$$

defines a probability density on $\mathbb{Z}^2$ and that

$$\widehat{p}_{t,\varepsilon}\Big(\frac{\lambda}{\sqrt{t}}\Big) = \Lambda_\varepsilon(t)^{-1} \sum_{x \in \mathbb{Z}^2} h_\varepsilon\Big(\frac{x}{\sqrt{t}}\Big) \exp\Big\{ i\lambda \cdot \frac{x}{\sqrt{t}} \Big\} \longrightarrow \widehat{h}_\varepsilon(\lambda) \tag{8.9}$$

uniformly on $[-M,M]^2$ as $t\to\infty$. Consequently the family

$$\psi_t(x,y)=\int_{[-M,M]^2}\Big|\widehat{p}_{t,\varepsilon}\Big(\frac{\lambda}{\sqrt{t}}\Big)\Big|^2\Big[\int_0^\beta e^{i\lambda\cdot x(s)}ds\Big]\Big[\int_0^\beta e^{-i\lambda\cdot y(s')}ds'\Big]\,d\lambda \tag{8.10}$$

are convergent continuous functionals on $D\Big([0,\beta],\mathbb{R}^2\Big)\otimes D\Big([0,\beta],\mathbb{R}^2\Big)$. Therefore

$$\begin{aligned}&\Bigg(\frac{1}{t^2}\int_{[-M,M]^2}\Big|\widehat{p}_{t,\varepsilon}\Big(\frac{\lambda}{\sqrt{t}}\Big)\Big|^2\\&\qquad\Big[\int_0^{\beta t}\exp\Big\{i\lambda\cdot\frac{Z(s)}{\sqrt{t}}\Big\}ds\Big]\Big[\int_0^{\beta t}\exp\Big\{-i\lambda\cdot\frac{Z'(s')}{\sqrt{t}}\Big\}ds'\Big]\,d\lambda,\\&\qquad\qquad\frac{(\log t)^2}{t}\big|Z[0,\beta t]\cap Z'[0,\beta t]\big|\Bigg)\\&=\Bigg(\int_{[-M,M]^2}\Big|\widehat{p}_{t,\varepsilon}\Big(\frac{\lambda}{\sqrt{t}}\Big)\Big|^2\\&\qquad\Big[\int_0^{\beta}\exp\Big\{i\lambda\cdot Z^{(t)}(s)\Big\}ds\Big]\Big[\int_0^{\beta}\exp\Big\{-i\lambda\cdot (Z')^{(t)}(s')\Big\}ds'\Big]\,d\lambda,\\&\qquad\qquad\frac{(\log t)^2}{t}\big|Z[0,\beta t]\cap Z'[0,\beta t]\big|\Bigg)\\&\xrightarrow{d}\Bigg(\int_{[-M,M]^2}|\widehat{h}_\varepsilon(\lambda)|^2\Big[\int_0^\beta e^{i\lambda\cdot W(s)}ds\Big]\Big[\int_0^\beta e^{-i\lambda\cdot W'(s')}ds'\Big]\,d\lambda,\\&\qquad\qquad(2\pi)^2\det(\Gamma)\alpha([0,\beta]^2)\Bigg).\end{aligned}\tag{8.11}$$

It follows from (5.30) that

$$\sup_t\mathbb{E}\exp\Big\{\theta\frac{\log t}{t}|Z[0,t]|\Big\}<\infty \tag{8.12}$$

for all $\theta>0$. We will show that uniformly in $\lambda\in[-M,M]^2$

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t^2}\mathbb{E}\Big|&\int_0^{\beta t}\exp\Big\{i\lambda\cdot\frac{Z(s)}{\sqrt{t}}\Big\}ds\\&-\frac{\log t}{2\pi\sqrt{\det(\Gamma)}}\sum_{x\in Z[0,\beta t]}\exp\Big\{i\lambda\cdot\frac{x}{\sqrt{t}}\Big\}\Big|^2=0.\end{aligned}\tag{8.13}$$

Using the inequality

$$|AA' - BB'| \le |A(B - B')| + |(A - B)B'|,$$

the Cauchy-Schwarz inequality and (8.12), we see from (8.13) that uniformly in $\lambda \in [-M, M]^2$

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t^2}\mathbb{E}\bigg|\bigg[\int_0^{\beta t}\exp\Big\{i\lambda\cdot\frac{Z(s)}{\sqrt{t}}\Big\}ds\bigg]\bigg[\int_0^{\beta t}\exp\Big\{-i\lambda\cdot\frac{Z'(s')}{\sqrt{t}}\Big\}ds'\bigg] \\
&\qquad -\bigg(\frac{\log t}{2\pi\sqrt{\det(\Gamma)}}\bigg)^2\bigg[\sum_{x\in Z[0,\beta t]}\exp\Big\{i\lambda\cdot\frac{x}{\sqrt{t}}\Big\}\bigg] \\
&\qquad\qquad\bigg[\sum_{x\in Z'[0,\beta t]}\exp\Big\{i\lambda\cdot\frac{x'}{\sqrt{t}}\Big\}\bigg]\bigg| \\
&\qquad = 0.
\end{aligned} \tag{8.14}$$

Together with (8.11) this shows that

$$\begin{aligned}
&\Bigg(\Big(\frac{\log t}{2\pi t}\Big)^2\int_{[-M,M]^2}\Big|\widehat{p}_{t,\varepsilon}\Big(\frac{\lambda}{\sqrt{t}}\Big)\Big|^2 \\
&\qquad\bigg[\sum_{x\in Z[0,\beta t]}\exp\Big\{i\lambda\cdot\frac{x}{\sqrt{t}}\Big\}\bigg]\bigg[\sum_{x'\in Z'[0,\beta t]}\exp\Big\{-i\lambda\cdot\frac{x'}{\sqrt{t}}\Big\}\bigg]d\lambda, \\
&\qquad\qquad\frac{(\log t)^2}{t}\big|Z[0,\beta t]\cap Z'[0,\beta t]\big|\Bigg) \\
&\xrightarrow{d}\Bigg(\det(\Gamma)\int_{[-M,M]^2}|\widehat{h}_\varepsilon(\lambda)|^2\bigg[\int_0^\beta e^{i\lambda\cdot W(s)}ds\bigg]\bigg[\int_0^\beta e^{-i\lambda\cdot W'(s')}ds'\bigg]d\lambda, \\
&\qquad\qquad (2\pi)^2\det(\Gamma)\alpha([0,\beta]^2)\Bigg).
\end{aligned} \tag{8.15}$$

Notice by (8.9) that for any $\delta > 0$, one can take $M > 0$ sufficiently large so that

$$\Big|\widehat{p}_{t,\varepsilon}\Big(\frac{\lambda}{\sqrt{t}}\Big)\Big| < \delta, \quad \lambda \in [-\sqrt{t}\pi, \sqrt{t}\pi]^2 \setminus [M, M]^2, \tag{8.16}$$

if t is sufficiently large. Consequently

$$(8.17)\quad H_t =: \Big|\int_{[-\sqrt{t}\pi,\sqrt{t}\pi]^2\setminus[-M,M]^2} \Big|\widehat{p}_{t,\varepsilon}\Big(\frac{\lambda}{\sqrt{t}}\Big)\Big|^2 \Big[\sum_{x\in Z[0,\beta t]} \exp\Big\{i\lambda\cdot\frac{x}{\sqrt{t}}\Big\}\Big] \times\Big[\sum_{x'\in Z'[0,\beta t]} \exp\Big\{-i\lambda\cdot\frac{x'}{\sqrt{t}}\Big\}\Big]\,d\lambda\Big| \le (2\pi)^2\delta t\big|Z[0,\beta t]\cap Z'[0,\beta t]\big|.$$

It follows from (7.1) that $\Big(\log t/(2\pi t)\Big)^2 H_t \to 0$ in L^1 uniformly in large t as $M\to\infty$. Therefore, using (8.15) and the fact that $\widehat{h}\in L^2$, we obtain

$$\bigg(\Big(\frac{\log t}{2\pi t}\Big)^2\int_{[-\sqrt{t}\pi,\sqrt{t}\pi]^2} \Big|\widehat{p}_{t,\varepsilon}\Big(\frac{\lambda}{\sqrt{t}}\Big)\Big|^2 \Big[\sum_{x\in Z[0,\beta t]} \exp\Big\{i\lambda\cdot\frac{x}{\sqrt{t}}\Big\}\Big] \times\Big[\sum_{x'\in Z'[0,\beta t]} \exp\Big\{-i\lambda\cdot\frac{x'}{\sqrt{t}}\Big\}\Big]\,d\lambda,\ \ \frac{(\log t)^2}{t}\big|Z[0,\beta t]\cap Z'[0,\beta t]\big|\bigg)$$

$$\xrightarrow{d}\bigg(\det(\Gamma)\int_{\mathbb{R}^2}|\widehat{h}_\varepsilon(\lambda)|^2\Big[\int_0^\beta e^{i\lambda\cdot W(s)}ds\Big]\Big[\int_0^\beta e^{-i\lambda\cdot W'(s')}ds'\Big]\,d\lambda,$$

$$(8.18)\qquad (2\pi)^2\det(\Gamma)\alpha([0,\beta]^2)\bigg).$$

Note that

$$B_{t,\beta}(\varepsilon) = \sum_{x\in\mathbb{Z}^2}\Big[\sum_{y\in Z[0,\beta t]}\Lambda_\varepsilon(t)^{-1}h_\varepsilon\Big(\frac{x-y}{\sqrt{t}}\Big)\Big]\Big[\sum_{y'\in Z'[0,\beta t]}\Lambda_\varepsilon(t)^{-1}h_\varepsilon\Big(\frac{x-y'}{\sqrt{t}}\Big)\Big]$$

$$(8.19)\qquad = \sum_{y\in Z[0,\beta t]}\ \sum_{y'\in Z'[0,\beta t]}\Big[\sum_{x\in\mathbb{Z}^2} p_{t,\varepsilon}(x-y)p_{t,\varepsilon}(x-y')\Big].$$

It then follows from Parseval's identity that

$$(8.20)\quad (2\pi)^2 t\, B_{t,\beta}(\varepsilon) = t\int_{[-\pi,\pi]^2}|\widehat{p}_{t,\varepsilon}(\lambda)|^2\Big[\sum_{y\in Z[0,\beta t]} e^{i\lambda\cdot y}\Big]\Big[\sum_{y'\in Z'[0,\beta t]} e^{-i\lambda\cdot y'}\Big]\,d\lambda$$

$$= \int_{[-\sqrt{t}\pi,\sqrt{t}\pi]^2}\Big|\widehat{p}_{t,\varepsilon}\Big(\frac{\lambda}{\sqrt{t}}\Big)\Big|^2 \Big[\sum_{y\in Z[0,\beta t]} \exp\Big\{i\lambda\cdot\frac{y}{\sqrt{t}}\Big\}\Big]\Big[\sum_{y'\in Z'[0,\beta t]} \exp\Big\{-i\lambda\cdot\frac{y'}{\sqrt{t}}\Big\}\Big]\,d\lambda.$$

Similarly, using the fact that h_ε is symmetric so that $\widehat{h}_\varepsilon(\lambda)$ is real

$$\int_{\mathbb{R}^2} |\widehat{h}_\varepsilon(\lambda)|^2 \Big[\int_0^\beta e^{i\lambda\cdot W(s)} ds\Big]\Big[\int_0^\beta e^{-i\lambda\cdot W'(s')} ds'\Big]\, d\lambda = \alpha_\varepsilon([0,\beta]^2). \tag{8.21}$$

Thus, we have proved

$$\begin{aligned}\Big(\frac{(\log t)^2}{t} B_{t,\beta}(\varepsilon),\quad \frac{(\log t)^2}{t}\big|Z[0,\beta t]\cap Z'[0,\beta t]\big|\Big)& \\ \xrightarrow{d} \Big((2\pi)^2\det(\Gamma)\alpha_\varepsilon([0,\beta]^2),\quad (2\pi)^2\det(\Gamma)\alpha([0,\beta]^2)\Big)&.\end{aligned} \tag{8.22}$$

(8.5) follows from this.

Thus to complete the proof of (8.5) it only remains to show (8.13) uniformly in $\lambda\in[-M,M]^2$. We will show that for any $\delta>0$ we can find $\delta'>0$ and $t_0<\infty$ such that

$$\frac{1}{t^2}\mathbb{E}\Big|\int_0^{\beta t}\exp\Big\{i\lambda\cdot\frac{Z(s)}{\sqrt{t}}\Big\}ds - \int_0^{\beta t}\exp\Big\{i\gamma\cdot\frac{Z(s)}{\sqrt{t}}\Big\}ds\Big|^2 < \delta \tag{8.23}$$

and

$$\Big(\frac{\log t}{t}\Big)^2\mathbb{E}\Big|\sum_{x\in Z[0,\beta t]}\exp\Big\{i\lambda\cdot\frac{x}{\sqrt{t}}\Big\} - \sum_{x\in Z[0,\beta t]}\exp\Big\{i\gamma\cdot\frac{x}{\sqrt{t}}\Big\}\Big|^2 < \delta \tag{8.24}$$

for all $t\ge t_0$ and $|\lambda-\gamma|\le\delta'$. We then cover $[-M,M]^2$ by a finite number of discs $B(\lambda_k,\delta')$ of radius δ' centered at λ_k, $k=1,\dots,N$. Define $\tau(\lambda)=\lambda_k$ where k is the smallest integer with $\lambda\in B(\lambda_k,\delta')$. It follows from [8, (4.11)] that for a planar random walk S_n satisfying the assumptions of Theorem 1.5 and a bounded continuous function f on $\mathbb{R}^d$,

$$\frac{1}{n^2}\mathbb{E}\Big[\sum_{k=1}^n f\Big(\frac{S_k}{\sqrt{n}}\Big) - \frac{\log n}{2\pi\sqrt{\det(\Gamma)}}\sum_{x\in S[1,n]} f\Big(\frac{x}{\sqrt{n}}\Big)\Big]^2 \longrightarrow 0$$

as $n\to\infty$. This treatment used in [8] can be easily modified so the result can be extended to the Lévy process setting. Thus,

$$\frac{1}{t^2}\mathbb{E}\Big[\int_0^t f\Big(\frac{Z(s)}{\sqrt{t}}\Big)ds - \frac{\log t}{2\pi\sqrt{\det(\Gamma)}}\sum_{x\in Z[0,t]} f\Big(\frac{x}{\sqrt{t}}\Big)\Big]^2 \longrightarrow 0.$$

In particular, we can choose $t_1 < \infty$ such that for all $t \geq t_1$ and $k = 1, \ldots, N$,

(8.25)
$$\frac{1}{t^2}\mathbb{E}\Big|\int_0^{\beta t} \exp\Big\{i\lambda_k \cdot \frac{Z(s)}{\sqrt{t}}\Big\} ds - \frac{\log t}{2\pi\sqrt{\det(\Gamma)}} \sum_{x \in Z[0,\beta t]} \exp\Big\{i\lambda_k \cdot \frac{x}{\sqrt{t}}\Big\}\Big|^2 \leq \delta.$$

Hence, uniformly in $\lambda \in [-M, M]^2$ we have that for all $t \geq t_0 \vee t_1$

(8.26)
$$\frac{1}{t^2}\mathbb{E}\Big|\int_0^{\beta t} \exp\Big\{i\lambda \cdot \frac{Z(s)}{\sqrt{t}}\Big\} ds - \frac{\log t}{2\pi\sqrt{\det(\Gamma)}} \sum_{x \in Z[0,\beta t]} \exp\Big\{i\lambda \cdot \frac{x}{\sqrt{t}}\Big\}\Big|^2 \leq 3\delta$$

proving that (8.13) holds uniformly in $\lambda \in [-M, M]^2$.

(8.23) actually holds uniformly in t. To see this note that

$$\begin{aligned}
(8.27) \qquad & \frac{1}{t^2}\mathbb{E}\Big|\int_0^{\beta t} \exp\Big\{i\lambda \cdot \frac{Z(s)}{\sqrt{t}}\Big\} ds - \int_0^{\beta t} \exp\Big\{i\gamma \cdot \frac{Z(s)}{\sqrt{t}}\Big\} ds\Big|^2 \\
& \leq \frac{1}{t^2}\mathbb{E}\Big|\int_0^{\beta t} |\lambda - \gamma| \frac{|Z(s)|}{\sqrt{t}} ds\Big|^2 \\
& = \frac{|\lambda - \gamma|^2}{t^3}\mathbb{E}\int_0^{\beta t}\int_0^{\beta t} |Z(s)||Z(r)|\, ds\, dr \\
& \leq C\frac{|\lambda - \gamma|^2}{t^3}\int_0^{\beta t}\int_0^{\beta t} s^{1/2} r^{1/2}\, ds\, dr \leq C'|\lambda - \gamma|^2.
\end{aligned}$$

As for (8.24),

$$\begin{aligned}
(8.28) \qquad & \mathbb{E}\Big|\sum_{x \in Z[0,\beta t]} \exp\Big\{i\lambda \cdot \frac{x}{\sqrt{t}}\Big\} - \sum_{x \in Z[0,\beta t]} \exp\Big\{i\gamma \cdot \frac{x}{\sqrt{t}}\Big\}\Big|^2 \\
& \leq 4\mathbb{E}\Big\{|Z[0, \beta t]|^2 1_{\{\sup_{s \leq \beta t} |Z(s)| \geq C\sqrt{t}\}}\Big\} \\
& + |\lambda - \gamma|^2 \mathbb{E}\left\{\Big|\sum_{x \in Z[0,\beta t]} \frac{|x|}{\sqrt{t}}\Big|^2 1_{\{\sup_{s \leq \beta t} |Z(s)| \leq C\sqrt{t}\}}\right\} \\
& \leq 4\mathbb{E}\Big\{|Z[0, \beta t]|^2 1_{\{\sup_{s \leq \beta t} |Z(s)| \geq C\sqrt{t}\}}\Big\} + C^2|\lambda - \gamma|^2 \mathbb{E}|Z[0, \beta t]|^2
\end{aligned}$$

and by (8.12)

$$
\begin{aligned}
(8.29)\quad & 4\mathbb{E}\left\{|Z[0,\beta t]|^2 1_{\{\sup_{s\le\beta t}|Z(s)|\ge C\sqrt{t}\}}\right\} + C^2|\lambda-\gamma|^2\mathbb{E}|Z[0,\beta t]|^2 \\
&\le 4\left\{\mathbb{E}(|Z[0,\beta t]|^4)P(\sup_{s\le\beta t}|Z(s)|\ge C\sqrt{t})\right\}^{1/2} + C^2|\lambda-\gamma|^2\mathbb{E}|Z[0,\beta t]|^2 \\
&\le \left(\frac{ct}{\log t}\right)^2\left(4\left\{P(\sup_{s\le\beta t}|Z(s)|\ge C\sqrt{t})\right\}^{1/2} + C^2|\lambda-\gamma|^2\right).
\end{aligned}
$$

Taking C large and then choosing $\delta' > 0$ sufficiently small completes the proof of (8.24) and hence of (8.5).

We now prove (8.6). Using the facts that $\Lambda_\varepsilon(t) \sim t$, that

$$
\begin{aligned}
&\frac{1}{t}\sum_{x\in\mathbb{Z}^2}\Big[\sum_{y\in Z[0,\beta t]} h_\varepsilon\Big(\frac{x-y}{\sqrt{t}}\Big)\Big]^2 - \int_{\mathbb{R}^2}\Big[\sum_{y\in Z[0,\beta t]} h_\varepsilon\Big(x-\frac{y}{\sqrt{t}}\Big)\Big]^2 dx \\
(8.30)\qquad &\qquad\qquad\qquad\qquad\qquad\qquad = o(1)|Z[0,\beta t]|^2,
\end{aligned}
$$

(where the boundedness and continuity of h_ε is used), and (8.12) we need only show that

$$
\begin{aligned}
(8.31)\qquad & \Big(\frac{\log t}{t}\Big)^2\int_{\mathbb{R}^2}\Big[\sum_{y\in Z[0,\beta t]} h_\varepsilon\Big(x-\frac{y}{\sqrt{t}}\Big)\Big]^2 dx \\
&\qquad \xrightarrow{d} (2\pi)^2\det(\Gamma)\int_{\mathbb{R}^2}\Big(\int_0^\beta h_\varepsilon\big(W(s)-x\big)ds\Big)^2 dx.
\end{aligned}
$$

By the Parseval identity,

$$
\begin{aligned}
(8.32)\qquad & \int_{\mathbb{R}^2}\Big[\sum_{y\in Z[0,\beta t]} h_\varepsilon\Big(x-\frac{y}{\sqrt{t}}\Big)\Big]^2 dx \\
&= (2\pi)^{-2}\int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2} e^{i\lambda\cdot x}\sum_{y\in Z[0,\beta t]} h_\varepsilon\Big(x-\frac{y}{\sqrt{t}}\Big)\,dx\Big|^2 d\lambda \\
&= (2\pi)^{-2}\int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2} h_\varepsilon(x)e^{i\lambda\cdot x}dx\Big|^2\Big|\sum_{y\in Z[0,\beta t]}\exp\Big\{i\lambda\cdot\frac{y}{\sqrt{t}}\Big\}\Big|^2 d\lambda \\
&= \int_{\mathbb{R}^2}\big|\widehat{h}_\varepsilon(\lambda)\big|^2\Big|\sum_{y\in Z[0,\beta t]}\exp\Big\{i\lambda\cdot\frac{y}{\sqrt{t}}\Big\}\Big|^2 d\lambda.
\end{aligned}
$$

Let $M > 0$ be fixed and $\lambda_1, \cdots, \lambda_N$ and τ be defined as above. By [8, Theorem 7],

$$
\frac{\log t}{t}\Big(\sum_{y\in Z[0,\beta t]}\exp\Big\{i\lambda_1\cdot\frac{y}{\sqrt{t}}\Big\},\cdots,\sum_{y\in Z[0,\beta t]}\exp\Big\{i\lambda_N\cdot\frac{y}{\sqrt{t}}\Big\}\Big)
\xrightarrow{d}(2\pi)\sqrt{\det(\Gamma)}\Big(\int_0^\beta e^{i\lambda_1\cdot W(s)}ds,\cdots,\int_0^\beta e^{i\lambda_N\cdot W(s)}ds\Big). \tag{8.33}
$$

In particular,

$$
\begin{aligned}
&\Big(\frac{\log t}{t}\Big)^2\int_{[-M,M]^2}|\widehat{h}_\varepsilon(\lambda)|^2\Big|\sum_{y\in Z[0,\beta t]}\exp\Big\{i\tau(\lambda)\cdot\frac{y}{\sqrt{t}}\Big\}\Big|^2d\lambda \\
&=\sum_{k=1}^N\int_{B_k}|\widehat{h}_\varepsilon(\lambda)|^2\Big|\frac{\log t}{t}\sum_{y\in Z[0,\beta t]}\exp\Big\{i\lambda_k\cdot\frac{y}{\sqrt{t}}\Big\}\Big|^2d\lambda \\
&\xrightarrow{d}(2\pi)^2\det(\Gamma)\sum_{k=1}^N\int_{B_k}|\widehat{h}_\varepsilon(\lambda)|^2\Big|\int_0^\beta e^{i\lambda_k\cdot W(s)}ds\Big|^2d\lambda \\
&=(2\pi)^2\det(\Gamma)\int_{[-M,M]^2}|\widehat{h}_\varepsilon(\lambda)|^2\Big|\int_0^\beta e^{i\tau(\lambda)\cdot W(s)}ds\Big|^2d\lambda.
\end{aligned} \tag{8.34}
$$

Notice that the right hand side of (8.34) converges to

$$
(2\pi)^2\det(\Gamma)\int_{[-M,M]^2}|\widehat{h}_\varepsilon(\lambda)|^2\Big|\int_0^\beta e^{i\lambda\cdot W(s)}ds\Big|^2d\lambda \tag{8.35}
$$

as $N\to\infty$. Applying (8.24) to the left hand side of (8.34) gives

$$
\begin{aligned}
&\Big(\frac{\log t}{t}\Big)^2\int_{[-M,M]^2}|\widehat{h}_\varepsilon(\lambda)|^2\Big|\sum_{y\in Z[0,\beta t]}\exp\Big\{i\lambda\cdot\frac{y}{\sqrt{t}}\Big\}\Big|^2d\lambda \\
&\xrightarrow{d}(2\pi)^2\det(\Gamma)\int_{[-M,M]^2}|\widehat{h}_\varepsilon(\lambda)|^2\Big|\int_0^\beta e^{i\lambda\cdot W(s)}ds\Big|^2d\lambda.
\end{aligned} \tag{8.36}
$$

As $M\to\infty$, the right hand side of (8.36) converges to

$$
\begin{aligned}
&(2\pi)^2\det(\Gamma)\int_{\mathbb{R}^2}|\widehat{h}_\varepsilon(\lambda)|^2\Big|\int_0^\beta e^{i\lambda\cdot W(s)}ds\Big|^2d\lambda \\
&=\det(\Gamma)\int_{\mathbb{R}^2}\Big(\int_0^\beta h_\epsilon\big(W(s)-x\big)ds\Big)^2dx
\end{aligned} \tag{8.37}
$$

by Parseval's identity. Note

$$H'_{t,M} =: \int_{\mathbb{R}^2\setminus[-M,M]^2} |\widehat{h}_\varepsilon(\lambda)|^2 \Big| \sum_{y\in Z[0,\beta t]} \exp\Big\{ i\lambda\cdot \frac{y}{\sqrt{t}}\Big\}\Big|^2 d\lambda \tag{8.38}$$
$$\le \big|Z[0,\beta t]\big|^2 \int_{\mathbb{R}^2\setminus[-M,M]^2} |\widehat{h}_\varepsilon(\lambda)|^2 d\lambda.$$

It follows from (8.12) and the fact that $\widehat{h}_\varepsilon \in L^2$ that $\Big(\frac{\log t}{2\pi t}\Big)^2 H'_{t,M} \to 0$ in L^1 as $M\to\infty$ uniformly in t. Therefore, using the last three displays, we obtain

$$\Big(\frac{\log t}{t}\Big)^2 \int_{\mathbb{R}^2} |\widehat{h}_\varepsilon(\lambda)|^2 \Big| \sum_{y\in Z[0,\beta t]} \exp\Big\{ i\lambda\cdot \frac{y}{\sqrt{t}}\Big\}\Big|^2 d\lambda \tag{8.39}$$
$$\stackrel{d}{\longrightarrow} \det(\Gamma)\int_{\mathbb{R}^2}\Big(\int_0^\beta h_\epsilon\big(W(s)-x\big)ds\Big)^2 dx.$$

□

Proof of Lemma 7.7: Let $T>0$ be fixed for the moment. Write $\gamma_t = t/[T^{-1}b_t]$. We have

$$\mathbb{E}\exp\bigg\{\theta\sqrt{\frac{b_t}{t}}(\log t)\bigg(\Lambda_\varepsilon\Big(\frac{t}{b_t}\Big)^{-2}\sum_{x\in\mathbb{Z}^2}\Big[\sum_{y\in Z[0,t]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y)\Big)\Big]^2\bigg)^{1/2}\bigg\}$$
$$\le \Bigg[\mathbb{E}\exp\bigg\{\theta\sqrt{\frac{b_t}{t}}(\log t)$$
$$\times\bigg(\Lambda_\varepsilon\Big(\frac{t}{b_t}\Big)^{-2}\sum_{x\in\mathbb{Z}^2}\Big[\sum_{y\in Z[0,\gamma_t]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y)\Big)\Big]^2\bigg)^{1/2}\bigg\}\Bigg]^{[T^{-1}b_t]}. \tag{8.40}$$

We obtain from Lemma 8.1 (with t being replaced by t/b_t and $\beta = T$)

$$\frac{b_t}{t}(\log t)^2\Lambda_\varepsilon\Big(\frac{t}{b_t}\Big)^{-2}\sum_{x\in\mathbb{Z}^2}\Big[\sum_{y\in Z[0,\gamma_t]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y)\Big)\Big]^2$$
$$\stackrel{d}{\longrightarrow} (2\pi)^2\det(\Gamma)\int_{\mathbb{R}^2}\Big(\int_0^T h_\varepsilon\big(W(s)-x\big)ds\Big)^2 dx, \quad t\to\infty. \tag{8.41}$$

In addition,

$$\frac{b_t}{t}(\log t)^2\Lambda_\varepsilon\Big(\frac{t}{b_t}\Big)^{-2}\sum_{x\in\mathbb{Z}^2}\Big[\sum_{y\in Z[0,\gamma_t]}h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y)\Big)\Big]^2$$

$$\le\frac{b_t}{t}(\log t)^2\Lambda_\varepsilon\Big(\frac{t}{b_t}\Big)^{-2}||h||_\infty|Z[0,\gamma_t]|\sum_{\substack{x\in\mathbb{Z}^2\\ y\in Z[0,\gamma_t]}}h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y)\Big)$$

$$=\frac{b_t}{t}(\log t)^2\Lambda_\varepsilon\Big(\frac{t}{b_t}\Big)^{-2}||h||_\infty|Z[0,\gamma_t]|^2\sum_{x\in\mathbb{Z}^2}h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}x\Big)$$

$$\le C\Big(\frac{b_t}{t}\Big)^2(\log t)^2|Z[0,\gamma_t]|^2, \tag{8.42}$$

where in the last step we used (7.17). (8.12) together with (8.41) then implies that

$$\mathbb{E}\exp\Big\{\theta\sqrt{\tfrac{b_t}{t}}\Lambda_\varepsilon\Big(\tfrac{t}{b_t}\Big)^{-1}(\log t)\Big(\textstyle\sum_{x\in\mathbb{Z}^2}\Big[\sum_{y\in Z[0,\gamma_t]}h_\varepsilon\Big(\sqrt{\tfrac{b_t}{t}}(x-y)\Big)\Big]^2\Big)^{1/2}\Big\}$$

$$\longrightarrow\mathbb{E}\exp\Big\{2\pi\theta\sqrt{\det(\Gamma)}\Big(\textstyle\int_{\mathbb{R}^2}\Big(\int_0^T h_\varepsilon\big(W(s)-x\big)ds\Big)^2dx\Big)^{1/2}\Big\}. \tag{8.43}$$

Combining (8.40) and (8.43) we see that

$$\limsup_{t\to\infty}\frac{1}{b_t}\log\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_t}{t}}(\log t)|A_t(\varepsilon)|^{1/2}\Big\}$$

$$\le\frac{1}{T}\log\mathbb{E}\exp\Big\{2\pi\theta\sqrt{\det(\Gamma)}\Big(\int_{\mathbb{R}^2}\Big(\int_0^T h_\varepsilon\big(W(s)-x\big)ds\Big)^2dx\Big)^{1/2}\Big\}. \tag{8.44}$$

Then the upper bound for (7.19) follows from the fact that

$$\lim_{T\to\infty}\frac{1}{T}\log\mathbb{E}\exp\Big\{2\pi\theta\sqrt{\det(\Gamma)}\Big(\int_{\mathbb{R}^2}\Big(\int_0^T h_\varepsilon\big(W(s)-x\big)ds\Big)^2dx\Big)^{1/2}\Big\}$$

$$=\sup_{g\in\mathcal{F}}\Big\{2\pi\theta\sqrt{\det(\Gamma)}\Big(\int_{\mathbb{R}^2}|(g^2*h_\varepsilon)(x)|^2dx\Big)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x),\Gamma\nabla g(x)\rangle dx\Big\}.$$

This is [10, Theorem 7]. (Or see the earlier [7, Theorem 3.1], which uses a slightly different smoothing).

We now prove the lower bound for (7.19). Let f be a smooth function on $\mathbb{R}^2$ with compact support and

$$||f||_2 = \Big(\int_{\mathbb{R}^2} |f(x)|^2 dx \Big)^{1/2} = 1. \tag{8.45}$$

We can write

$$\begin{aligned} &\sqrt{\frac{b_t}{t}} \Big(\sum_{x\in\mathbb{Z}^2} \Big[\sum_{y\in Z[0,t]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y)\Big)\Big]^2\Big)^{1/2} \\ &= \sqrt{\frac{b_t}{t}} \Big(\int_{\mathbb{R}^2} \Big[\sum_{y\in Z[0,t]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}([x]-y)\Big)\Big]^2 dx\Big)^{1/2} \\ &= \Big(\int_{\mathbb{R}^2} \Big[\sum_{y\in Z[0,t]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(\Big[\sqrt{\frac{t}{b_t}}x\Big]-y)\Big)\Big]^2 dx\Big)^{1/2}. \end{aligned} \tag{8.46}$$

Hence by the Cauchy-Schwarz inequality,

$$\begin{aligned} &\sqrt{\frac{b_t}{t}} \Big(\sum_{x\in\mathbb{Z}^2} \Big[\sum_{y\in Z[0,t]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y)\Big)\Big]^2\Big)^{1/2} \\ &= \Big(\int_{\mathbb{R}^2} \Big[\sum_{y\in Z[0,t]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}\Big[\sqrt{\frac{t}{b_t}}x\Big]-\sqrt{\frac{b_t}{t}}y\Big)\Big]^2 dx\Big)^{1/2} \\ &\geq \int_{\mathbb{R}^2} f(x) \sum_{y\in Z[0,t]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}\Big[\sqrt{\frac{t}{b_t}}x\Big]-\sqrt{\frac{b_t}{t}}y\Big) dx \end{aligned}$$

$$= \int_{\mathbb{R}^2} f(x) \sum_{y\in Z[0,t]} h_\varepsilon\Big(x-\sqrt{\frac{b_t}{t}}y\Big) dx + O(1)|Z[0,t]|, \quad t\to\infty, \tag{8.47}$$

where $O(1)$ is bounded by a constant. In view of (4.12), recalling that

$$\sqrt{\frac{b_t}{t}}|A_t(\varepsilon)|^{1/2} \sim \frac{b_t}{t}\sqrt{\frac{b_t}{t}} \Big(\sum_{x\in\mathbb{Z}^2} \Big[\sum_{y\in Z[0,t]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y)\Big)\Big]^2\Big)^{1/2},$$

and using Hölder's inequality one can see that the term $O(1)|Z[0,t]|$ does not contribute anything to (7.19).

By [8, Theorem 8],

(8.48)

$$\liminf_{t\to\infty}\frac{1}{b_t}\log\mathbb{E}\exp\Big\{\theta\frac{b_t\log t}{t}\sum_{y\in Z[0,t]}(f*h_\varepsilon)\Big(\sqrt{\frac{b_t}{t}}y\Big)\Big\}$$
$$\geq\sup_{g\in\mathcal{F}}\Big\{2\pi\theta\sqrt{\det(\Gamma)}\int_{\mathbb{R}^2}(f*h_\varepsilon)(x)g^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x),\Gamma\nabla g(x)\rangle dx\Big\}$$
$$=\sup_{g\in\mathcal{F}}\Big\{2\pi\theta\sqrt{\det(\Gamma)}\int_{\mathbb{R}^2}f(x)(g^2*h_\varepsilon)(x)dx-\frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x),\Gamma\nabla g(x)\rangle dx\Big\}.$$

We see from (8.47) and (8.48) that

$$\liminf_{t\to\infty}\frac{1}{b_t}\log\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_t}{t}}(\log t)|A_t(\varepsilon)|^{1/2}\Big\}\tag{8.49}$$
$$\geq\sup_{g\in\mathcal{F}}\Big\{2\pi\theta\sqrt{\det(\Gamma)}\int_{\mathbb{R}^2}f(x)(g^2*h_\varepsilon)(x)dx$$
$$-\frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x),\Gamma\nabla g(x)\rangle dx\Big\}.$$

Taking the supremum over f on the right gives

$$\liminf_{t\to\infty}\frac{1}{b_t}\log\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_t}{t}}(\log t)|A_t(\varepsilon)|^{1/2}\Big\}\tag{8.50}$$
$$\geq\sup_{g\in\mathcal{F}}\Big\{2\pi\theta\sqrt{\det(\Gamma)}\Big(\int_{\mathbb{R}^2}|(g^2*h_\varepsilon)(x)|^2dx\Big)^{1/2}$$
$$-\frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x),\Gamma\nabla g(x)\rangle dx\Big\}.$$

This completes the proof of (7.19).

To prove (7.20), in (7.19) we replace t by $2^{-N}t$, θ by $2^{-N/2}\theta$, b_t by $\widetilde{b}_t =: b_{2^N t}$

and ε by $2^{N/2}\varepsilon$ to find that

(8.51)

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{b_t}\log\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_t}{t}}(\log t)\\
&\qquad\times\Big(\Lambda_\varepsilon\Big(\frac{t}{b_t}\Big)^{-2}\sum_{x\in\mathbb{Z}^2}\Big[\sum_{y\in Z[0,2^{-N}t]}h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y)\Big)\Big]^2\Big)^{1/2}\Big\}\\
&\quad=\lim_{t\to\infty}\frac{1}{\widetilde{b}_{2^{-N}t}}\log\mathbb{E}\exp\Big\{2^{-N/2}\theta\sqrt{\frac{\widetilde{b}_{2^{-N}t}}{2^{-N}t}}(\log t)\\
&\qquad\times\Big(\Lambda_{2^{N/2}\varepsilon}\Big(\frac{2^{-N}t}{\widetilde{b}_{2^{-N}t}}\Big)^{-2}\sum_{x\in\mathbb{Z}^2}\Big[\sum_{y\in Z[0,2^{-N}t]}h_{2^{N/2}\varepsilon}\Big(\sqrt{\frac{\widetilde{b}_{2^{-N}t}}{2^{-N}t}}(x-y)\Big)\Big]^2\Big)^{1/2}\Big\}\\
&\quad=\sup_{g\in\mathcal{F}}\Big\{2\pi 2^{-N/2}\theta\sqrt{\det(\Gamma)}\Big(\int_{\mathbb{R}^2}|(g^2*h_{2^{N/2}\varepsilon})(x)|^2dx\Big)^{1/2}\\
&\qquad-\frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x),\Gamma\nabla g(x)\rangle^2dx\Big\}\\
&\quad\le\sup_{g\in\mathcal{F}}\Big\{2\pi 2^{-N/2}\theta\sqrt{\det(\Gamma)}\Big(\int_{\mathbb{R}^2}|g(x)|^4dx\Big)^{1/2}\\
&\qquad-\frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x),\Gamma\nabla g(x)\rangle^2dx\Big\}.\\
&\quad=\big(2\pi 2^{-N/2}\theta\big)^2\sqrt{\det(\Gamma)}\sup_{f\in\mathcal{F}}\Big\{\Big(\int_{\mathbb{R}^2}|f(x)|^4dx\Big)^{1/2}\\
&\qquad-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla f(x)|^2dx\Big\}.\\
&\quad=2^{-N+2}\pi^2\theta^2\sqrt{\det(\Gamma)}\kappa(2,2)^4,
\end{aligned}$$

where the third step follows from Jensen's inequality, the fourth step follows from the substitution $g(x)=\sqrt{|\det(A)|}f(Ax)$ with the 2×2 matrix A satisfying

$$A^\tau\Gamma A=\big(2\pi 2^{-N/2}\theta\big)^2\sqrt{\det(\Gamma)}I_{2\times 2},$$

and the last step follows from Lemma 7.2 in [7]. □

9 Exponential approximation

Let $t_1, \cdots, t_a \geq 0$ and write

$$\Delta_1 = [0, t_1], \quad \text{and} \quad \Delta_k = \Big[\sum_{j=1}^{k-1} t_j, \ \sum_{j=1}^{k} t_j\Big] \quad k = 2, \cdots, a. \tag{9.1}$$

Let $p(x)$ be a positive symmetric function on $\mathbb{Z}^2$ with $\sum_{x\in\mathbb{Z}^2} p(x) = 1$ and define

$$L = \sum_{j,k=1}^{a} \Big[\big|Z(\Delta_j) \cap Z'(\Delta_k)\big| - \sum_{x\in\mathbb{Z}^2} p(x)\big|Z(\Delta_j) \cap \big(Z'(\Delta_k) + x\big)\big|\Big], \tag{9.2}$$

and

$$\begin{aligned} L_j = &\big|Z[0, t_j] \cap Z'[0, t_j]\big| \\ &- \sum_{x\in\mathbb{Z}^2} p(x)\big|Z[0, t_j] \cap \big(Z'[0, t_j] + x\big)\big|, \quad j = 1, \cdots, a. \end{aligned} \tag{9.3}$$

Lemma 9.1 *For any $m \geq 1$,*

$$\mathbb{E}L^m \geq 0 \tag{9.4}$$

and

$$\Big\{\mathbb{E}L^m\Big\}^{1/2} \leq \sum_{\substack{k_1+\cdots+k_a=m \\ k_1,\cdots,k_a\geq 0}} \frac{m!}{k_1!\cdots k_a!}\Big\{\mathbb{E}|L_1|^{k_1}\Big\}^{1/2} \cdots \Big\{\mathbb{E}|L_a|^{k_a}\Big\}^{1/2}. \tag{9.5}$$

Consequently, for any $\theta > 0$

$$\sum_{m=0}^{\infty} \frac{\theta^m}{m!}\Big\{\mathbb{E}L^m\Big\}^{1/2} \leq \prod_{j=1}^{a} \sum_{m=0}^{\infty} \frac{\theta^m}{m!}\Big\{\mathbb{E}|L_j|^m\Big\}^{1/2}. \tag{9.6}$$

Proof. Write

$$\widehat{p}(\lambda) = \sum_{x\in\mathbb{Z}^2} p(x) e^{i\lambda\cdot x}. \tag{9.7}$$

We note that

$$|\widehat{p}(\lambda)| \le \widehat{p}(0) = 1. \tag{9.8}$$

Notice also that

$$L = \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} \big[1 - \widehat{p}(\lambda)\big] \Big[\sum_{j=1}^{a} \sum_{x \in Z(\Delta_j)} e^{i\lambda\cdot x} \Big] \Big[\sum_{j'=1}^{a} \sum_{x' \in Z'(\Delta_{j'})} e^{-i\lambda\cdot x'} \Big] \, d\lambda. \tag{9.9}$$

We therefore have

$$\begin{aligned} \mathbb{E}L^m &= \frac{1}{(2\pi)^{2m}} \int_{([-\pi,\pi]^2)^m} \Big| \mathbb{E} \prod_{k=1}^{m} \sum_{j=1}^{a} \sum_{x_k \in Z(\Delta_j)} e^{i\lambda_k\cdot x_k} \Big|^2 \Big(\prod_{k=1}^{m} \big[1 - \widehat{p}(\lambda_k)\big] d\lambda_k \Big) \\ &= \frac{1}{(2\pi)^{2m}} \int_{([-\pi,\pi]^2)^m} \Big| \sum_{l_1,\cdots,l_m=1}^{a} \mathbb{E}\Big(H_{l_1}(\lambda_1) \cdots H_{l_m}(\lambda_m) \Big) \Big|^2 \\ &\qquad\qquad \Big(\prod_{k=1}^{m} \big[1 - \widehat{p}(\lambda_k)\big] d\lambda_k \Big), \end{aligned} \tag{9.10}$$

where

$$H_j(\lambda) = \sum_{x \in Z(\Delta_j)} e^{i\lambda\cdot x}. \tag{9.11}$$

This proves (9.4) and implies that

$$\begin{aligned} &\Big\{ \mathbb{E}L^m \Big\}^{1/2} \\ &\quad \le \frac{1}{(2\pi)^m} \sum_{l_1,\cdots,l_m=1}^{a} \Big\{ \int_{([-\pi,\pi]^2)^m} \Big| \mathbb{E}\Big(H_{l_1}(\lambda_1) \cdots H_{l_m}(\lambda_m) \Big) \Big|^2 \\ &\qquad\qquad \Big(\prod_{k=1}^{m} \big[1 - \widehat{p}(\lambda_k)\big] d\lambda_k \Big) \Big\}^{1/2}. \end{aligned} \tag{9.12}$$

Note that for any $k > j$ we can write

$$H_k(\lambda) = \sum_{x \in Z(\Delta_k)} e^{i\lambda\cdot x} = e^{i\lambda\cdot Z(t_j)} H_k^{(j)}(\lambda), \tag{9.13}$$

where

$$H_k^{(j)}(\lambda) = \sum_{x \in Z(\Delta_k) - Z(t_j)} e^{i\lambda \cdot x} \tag{9.14}$$

is independent of $\mathcal{F}_{t_j}$.

Let $1 \le l_1, \cdots, l_m \le a$ be fixed and let $k_j = \sum_{i=1}^m \delta(l_i, j)$ be the number of l's which are equal to j, for each $1 \le j \le a$. Then using independence

(9.15)

$$\begin{aligned}
&\int_{([-\pi,\pi]^2)^m} \Big|\mathbb{E}\Big(H_{l_1}(\lambda_1)\cdots H_{l_m}(\lambda_m)\Big)\Big|^2 \Big(\prod_{k=1}^m \big[1-\widehat{p}(\lambda_k)\big]\,d\lambda_k\Big) \\
&= \int_{([-\pi,\pi]^2)^m} \Big|\mathbb{E}\prod_{j=1}^a \Big(H_j(\lambda_{j,1})\cdots H_j(\lambda_{j,k_j})\Big)\Big|^2 \Big(\prod_{j=1}^a\prod_{l=1}^{k_j} \big[1-\widehat{p}(\lambda_{j,l})\big]\,d\lambda_{j,l}\Big) \\
&= \int_{([-\pi,\pi]^2)^m} \Big|\mathbb{E}\Big[\exp\Big\{i\Big(\sum_{j=2}^a\sum_{l=1}^{k_j}\lambda_{j,l}\Big)\cdot Z(t_1)\Big\} \times \Big(H_1(\lambda_{1,1})\cdots H_1(\lambda_{1,k_1})\Big)\Big] \\
&\qquad\qquad \mathbb{E}\Big(\prod_{j=2}^a \Big(H_j^{(1)}(\lambda_{j,1})\cdots H_j^{(1)}(\lambda_{j,k_j})\Big)\Big|^2 \Big(\prod_{j=1}^a\prod_{l=1}^{k_j} \big[1-\widehat{p}(\lambda_{j,l})\big]\,d\lambda_{j,l}\Big) \\
&= \int_{([-\pi,\pi]^2)^{m-k_1}} \Big|\mathbb{E}\Big(\prod_{j=2}^a \Big(H_j^{(1)}(\lambda_{j,1})\cdots H_j^{(1)}(\lambda_{j,k_j})\Big)\Big|^2 \\
&\qquad F(\lambda_{2,1},\cdots,\lambda_{2,k_2};\cdots;\lambda_{a,1},\cdots,\lambda_{a,k_a})\Big(\prod_{j=2}^a\prod_{l=1}^{k_j} \big[1-\widehat{p}(\lambda_{j,l})\big]\,d\lambda_{j,l}\Big),
\end{aligned}$$

where

(9.16)

$$\begin{aligned}
&F(\lambda_{2,1},\cdots,\lambda_{2,k_2};\cdots;\lambda_{a,1},\cdots,\lambda_{a,k_a}) \\
&= \int_{([-\pi,\pi]^2)^{k_1}} \Big|\mathbb{E}\Big[\exp\Big\{i\Big(\sum_{j=2}^a\sum_{l=1}^{k_j}\lambda_{j,l}\Big)\cdot Z(t_1)\Big\} \times \Big(H_1(\lambda_{1,1})\cdots H_1(\lambda_{1,k_1})\Big)\Big]\Big|^2 \\
&\qquad\qquad\qquad\qquad \Big(\prod_{l=1}^{k_1} \big[1-\widehat{p}(\lambda_{1,l})\big]\,d\lambda_{1,l}\Big).
\end{aligned}$$

Notice that by symmetry

$$\mathbb{E}\Big[\exp\Big\{i\Big(\sum_{j=2}^{a}\sum_{l=1}^{k_j}\lambda_{j,l}\Big)\cdot Z(t_1)\Big\}\Big(H_1(\lambda_{1,1})\cdots H_1(\lambda_{1,k_1})\Big)\Big] \tag{9.17}$$

is real valued. Hence if Z' denotes an independent copy of Z, and H_1' is obtained from H_1 by replacing Z by Z',

$$\begin{aligned}
&F(\lambda_{2,1},\cdots,\lambda_{2,k_2};\cdots;\lambda_{a,1},\cdots,\lambda_{a,k_a}) \\
&=\int_{([-\pi,\pi]^2)^{k_1}}\mathbb{E}\Big[\exp\Big\{i\Big(\sum_{j=2}^{a}\sum_{l=1}^{k_j}\lambda_{j,l}\Big)\cdot\big(Z(t_1)+Z'(t_1)\big)\Big\} \\
&\qquad\times\prod_{l=1}^{k_1}\Big(H_1(\lambda_{1,l})H_1'(\lambda_{1,l})\Big)\Big]\Big(\prod_{l=1}^{k_1}\big[1-\widehat{p}(\lambda_{1,l})\big]\,d\lambda_{1,l}\Big) \\
&=\mathbb{E}\Big[\exp\Big\{i\Big(\sum_{j=2}^{a}\sum_{l=1}^{k_j}\lambda_{j,l}\Big)\cdot\big(Z(t_1)+Z'(t_1)\big)\Big\} \\
&\qquad\times\int_{([-\pi,\pi]^2)^{k_1}}\prod_{l=1}^{k_1}\Big(H_1(\lambda_{1,l})H_1'(\lambda_{1,l})\Big)\Big]\Big(\prod_{l=1}^{k_1}\big[1-\widehat{p}(\lambda_{1,l})\big]\,d\lambda_{1,l}\Big).
\end{aligned} \tag{9.18}$$

By the fact that

$$\begin{aligned}
&\int_{([-\pi,\pi]^2)^{k_1}}\prod_{l=1}^{k_1}\Big(H_1(\lambda_{1,l})H_1'(\lambda_{1,l})\Big)\Big(\prod_{l=1}^{k_1}\big[1-\widehat{p}(\lambda_{1,l})\big]\,d\lambda_{1,l}\Big) \\
&=\Big[\int_{[-\pi,\pi]^2}\big[1-\widehat{p}(\lambda)\big]H_1(\lambda)H_1'(\lambda)\,d\lambda\Big]^{k_1}=(2\pi)^{2k_1}L_1^{k_1},
\end{aligned} \tag{9.19}$$

we have proved that

$$\begin{aligned}
&\int_{([-\pi,\pi]^2)^m}\Big|\mathbb{E}\Big(H_{l_1}(\lambda_1)\cdots H_{l_m}(\lambda_m)\Big)\Big|^2\Big(\prod_{k=1}^{m}\big[1-\widehat{p}(\lambda_k)\big]\,d\lambda_k\Big) \\
&\le(2\pi)^{2k_1}\mathbb{E}|L_1|^{k_1}\int_{([-\pi,\pi]^2)^{m-k_1}}\Big|\mathbb{E}\Big(\prod_{j=2}^{a}\Big(H_j^{(1)}(\lambda_{j,1})\cdots H_j^{(1)}(\lambda_{j,k_j})\Big)\Big|^2 \\
&\qquad\Big(\prod_{j=2}^{a}\prod_{l=1}^{k_j}\big[1-\widehat{p}(\lambda_{j,l})\big]\,d\lambda_{j,l}\Big).
\end{aligned} \tag{9.20}$$

Repeating the above procedure,

$$
\int_{([-\pi,\pi]^2)^m} \Big|\mathbb{E}\Big(H_{l_1}(\lambda_1)\cdots H_{l_m}(\lambda_m)\Big)\Big|^2 \Big(\prod_{k=1}^m \big[1-\widehat{p}(\lambda_k)\big]d\lambda_k\Big)
$$

$$
\le \prod_{j=1}^a \Big\{(2\pi)^{2k_j}\mathbb{E}|L_j|^{k_j}\Big\} = (2\pi)^{2m}\prod_{j=1}^a \mathbb{E}|L_j|^{k_j}. \tag{9.21}
$$

Our Lemma now follows from (9.12). □

Proof of Lemma 7.8: Define

$$
q_{t,\varepsilon}(x) = \Lambda_\varepsilon\Big(\frac{t}{b_t}\Big)^{-2}\sum_{z\in\mathbb{Z}^2} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-z)\Big)h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}z\Big), \qquad x\in\mathbb{Z}^2. \tag{9.22}
$$

Then $q_{t,\varepsilon}(x)$ is a probability density on $\mathbb{Z}^2$. We claim that

$$
B_t^{(0)}(\varepsilon) = \sum_{x\in\mathbb{Z}^2} q_{t,\varepsilon}(x)\big|Z[0,t]\cap\big(x+Z'[0,t]\big)\big|. \tag{9.23}
$$

This follows from the fact that

$$
\begin{aligned}
&\sum_{x\in\mathbb{Z}^2}\sum_{y\in Z[0,t]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y)\Big)\sum_{y'\in Z'[0,t]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y')\Big) \\
&= \sum_{x\in\mathbb{Z}^2}\sum_{y'\in Z'[0,t]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}x\Big)\sum_{y\in\mathbb{Z}^2} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x+y'-y)\Big)1_{\{y\in Z[0,t]\}} \\
&= \sum_{x\in\mathbb{Z}^2}\sum_{y'\in Z'[0,t]} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}x\Big)\sum_{y\in\mathbb{Z}^2} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y)\Big)1_{\{y+y'\in Z[0,t]\}} \\
&= \sum_{y\in\mathbb{Z}^2}\sum_{x\in\mathbb{Z}^2} h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}x\Big)h_\varepsilon\Big(\sqrt{\frac{b_t}{t}}(x-y)\Big)\big|Z'[0,t]\cap\big(Z[0,t]-y\big)\big|
\end{aligned} \tag{9.24}
$$

and

$$
\big|Z'[0,t]\cap\big(Z[0,t]-y\big)\big| = \big|Z[0,t]\cap\big(Z'[0,t]+y\big)\big|. \tag{9.25}
$$

Write $\gamma_t = t/[b_t]$ and $\Delta_j = [(j-1)\gamma_t, j\gamma_t]$, $j=1,\cdots,[b_t]$. Note that

$$
\begin{aligned}
&\textstyle\sum_{j=1}^{[b_t]}\big|Z(\Delta_j)\cap Z'[0,t]\big| - \sum_{1\le j<k\le[b_t]}\big|Z(\Delta_j)\cap Z(\Delta_k)\cap Z'[0,t]\big| \\
&\le \big|Z[0,t]\cap Z'[0,t]\big| \le \textstyle\sum_{j=1}^{[b_t]}\big|Z(\Delta_j)\cap Z'[0,t]\big|
\end{aligned} \tag{9.26}
$$

and similarly

$$\begin{aligned}(9.27)\qquad &\sum_{j=1}^{[b_t]}\sum_{x\in\mathbb{Z}^2} q_{t,\varepsilon}(x)\big|Z(\Delta_j)\cap\big(x+Z'[0,t]\big)\big| \\ &-\sum_{1\le j<k\le[b_t]}\sum_{x\in\mathbb{Z}^2} q_{t,\varepsilon}(x)\big|Z(\Delta_j)\cap Z(\Delta_k)\cap\big(x+Z'[0,t]\big)\big| \\ &\le \sum_{x\in\mathbb{Z}^2} q_{t,\varepsilon}(x)\big|Z[0,t]\cap\big(x+Z'[0,t]\big)\big| \\ &\le \sum_{j=1}^{[b_t]}\sum_{x\in\mathbb{Z}^2} q_{t,\varepsilon}(x)\big|Z(\Delta_j)\cap\big(x+Z'[0,t]\big)\big|.\end{aligned}$$

Hence,

$$\begin{aligned}&\bigg|\big|Z[0,t]\cap Z'[0,t]\big|-\sum_{x\in\mathbb{Z}^2} q_{t,\varepsilon}(x)\big|Z[0,t]\cap\big(x+Z'[0,t]\big)\big|\bigg| \\ &\le\bigg|\sum_{j=1}^{[b_t]}\Big[\big|Z(\Delta_j)\cap Z'[0,t]\big|-\sum_{x\in\mathbb{Z}^2} q_{t,\varepsilon}(x)\big|Z(\Delta_j)\cap\big(x+Z'[0,t]\big)\big|\Big]\bigg| \\ &+\sum_{1\le j<k\le[b_t]}\big|Z(\Delta_j)\cap Z(\Delta_k)\cap Z'[0,t]\big| \\ (9.28)\qquad &+\sum_{1\le j<k\le[b_t]}\sum_{x\in\mathbb{Z}^2} q_{t,\varepsilon}(x)\big|Z(\Delta_j)\cap Z(\Delta_k)\cap\big(x+Z'[0,t]\big)\big|.\end{aligned}$$

We first take care of the last two terms. This is the easy step. Write

$$\begin{aligned}(9.29)\qquad \eta(t,\varepsilon)=&\sum_{1\le j<k\le[b_t]}\big|Z(\Delta_j)\cap Z(\Delta_k)\cap Z'[0,t]\big| \\ &+\sum_{1\le j<k\le[b_t]}\sum_{x\in\mathbb{Z}^2} q_{t,\varepsilon}(x)\big|Z(\Delta_j)\cap Z(\Delta_k)\cap\big(x+Z'[0,t]\big)\big|.\end{aligned}$$

It follows from (7.2) that

$$(9.30)\qquad \sup_{t,j,k,x}\mathbb{E}\exp\left\{c\frac{(\log t)^{3/2}}{\sqrt{t}}\big|Z(\Delta_j)\cap Z(\Delta_k)\cap\big(x+Z'[0,t]\big)\big|^{1/2}\right\}<\infty$$

for some $c > 0$. Hence, if $b_t = o\big((\log t)^{1/5}\big)$, then for any $\theta > 0$ we can find $t_0 < \infty$ such that

(9.31)

$$\begin{aligned}
&\sup_{t\geq t_0}\mathbb{E}\exp\left\{\theta\sqrt{\frac{b_t}{t}}(\log t)\eta(t,\varepsilon)^{1/2}\right\}\\
&\qquad\leq \sup_{t\geq t_0}\sup_{j,k,x}\mathbb{E}\exp\left\{\theta\sqrt{\frac{b_t}{t}}(\log t)b_t^2\big|Z(\Delta_j)\cap Z(\Delta_k)\cap\big(x+Z'[0,t]\big)\big|^{1/2}\right\}\\
&\qquad< \infty.
\end{aligned}$$

Hence

$$\limsup_{t\to\infty}\frac{1}{b_t}\log\mathbb{E}\exp\left\{\theta\sqrt{\frac{b_t}{t}}(\log t)\eta(t,\varepsilon)^{1/2}\right\} = 0. \tag{9.32}$$

To handle the first term on the right hand side of (9.28) set

$$\xi(t,\varepsilon) = \sum_{j=1}^{[b_t]}\Big[\big|Z(\Delta_j)\cap Z'[0,t]\big| - \sum_{x\in\mathbb{Z}^2} q_{t,\varepsilon}(x)\big|Z(\Delta_j)\cap\big(x+Z'[0,t]\big)\big|\Big]. \tag{9.33}$$

Using Fubini, independence and then the Cauchy-Schwarz inequality we have

(9.34)

$$\begin{aligned}
&\big|\mathbb{E}\xi^m(t,\varepsilon)\big| = \\
&(2\pi)^{-2m}\bigg|\mathbb{E}\int_{([-\pi,\pi]^2)^m}\Big(\prod_{k=1}^m\big[1-\widehat{q}_{t,\varepsilon}(\lambda_k)\big]\Big)\\
&\quad\times\bigg[\prod_{k=1}^m\sum_{x'_k\in Z'[0,t]} e^{i\lambda_k\cdot x'_k}\bigg]\bigg[\prod_{k=1}^m\sum_{j=1}^{[b_n]}\sum_{x_k\in Z(\Delta_j)} e^{-i\lambda_k\cdot x_k}\bigg]\,d\lambda_1\cdots d\lambda_m\bigg|\\
&\leq (2\pi)^{-2m}\bigg\{\int_{([-\pi,\pi]^2)^m}\Big(\prod_{k=1}^m\big[1-\widehat{q}_{t,\varepsilon}(\lambda_k)\big]\Big)\bigg|\mathbb{E}\prod_{k=1}^m\sum_{x_k\in Z[0,t]} e^{i\lambda_k\cdot x_k}\bigg|^2 d\lambda_1\cdots d\lambda_m\bigg\}^{1/2}\\
&\times\bigg\{\int_{([-\pi,\pi]^2)^m}\Big(\prod_{k=1}^m\big[1-\widehat{q}_{t,\varepsilon}(\lambda_k)\big]\Big)\bigg|\mathbb{E}\prod_{k=1}^m\sum_{j=1}^{[b_t]}\sum_{x_k\in Z(\Delta_j)} e^{i\lambda_k\cdot x_k}\bigg|^2 d\lambda_1\cdots d\lambda_m\bigg\}^{1/2}\\
&\leq \Big\{\mathbb{E}|Z[0,t]\cap Z'[0,t]|^m\Big\}^{1/2}\Big\{\mathbb{E}\zeta^m(t,\varepsilon)\Big\}^{1/2},
\end{aligned}$$

where

$$(9.35)\quad \zeta(t,\varepsilon) = \sum_{j,k=1}^{[b_t]} \Big[\big|Z(\Delta_j)\cap Z'(\Delta_k)\big| - \sum_{x\in\mathbb{Z}^2} q_{t,\varepsilon}(x)\big|Z(\Delta_j)\cap\big(x+Z'(\Delta_k)\big)\big|\Big]$$

and we have used the fact that $1-\widehat{q}_{t,\varepsilon}(\lambda)\le 1$ in the last step. Note that in the notation of (9.2), $\zeta(t,\varepsilon)=L$ with $p(x)=q_{t,\varepsilon}(x)$, so that by (9.4), for all $m\ge 1$

$$(9.36)\qquad \mathbb{E}\zeta^m(t,\varepsilon)\ge 0.$$

Let $\delta>0$ be fixed for a while. By Cauchy-Schwarz and then (9.34)

$$\begin{aligned}(9.37)\qquad &\mathbb{E}\cosh\Big\{\theta\sqrt{\frac{b_t}{t}}(\log t)|\xi(t,\varepsilon)|^{1/2}\Big\}\\ &=\sum_{m=0}^\infty \frac{\theta^{2m}}{(2m)!}\Big(\sqrt{\frac{b_t}{t}}(\log t)\Big)^{2m}\mathbb{E}|\xi^m(t,\varepsilon)|\\ &\le \sum_{m=0}^\infty \frac{\theta^{2m}}{(2m)!}\Big(\sqrt{\frac{b_t}{t}}(\log t)\Big)^{2m}\Big\{\mathbb{E}\xi^{2m}(t,\varepsilon)\Big\}^{1/2}\\ &\le \Big\{\sum_{m=0}^\infty \frac{(\delta\theta)^{2m}}{(2m)!}\Big(\sqrt{\frac{b_t}{t}}(\log t)\Big)^{2m}\Big\{\mathbb{E}|Z[0,t]\cap Z'[0,t]|^{2m}\Big\}^{1/2}\Big\}^{1/2}\\ &\times\Big\{\sum_{m=0}^\infty \frac{(\delta^{-1}\theta)^{2m}}{(2m)!}\Big(\sqrt{\frac{b_t}{t}}(\log t)\Big)^{2m}\Big\{\mathbb{E}\zeta^{2m}(t,\varepsilon)\Big\}^{1/2}\Big\}^{1/2}\\ &\le \Big\{\sum_{m=0}^\infty \frac{(\delta\theta)^{m}}{m!}\Big(\sqrt{\frac{b_t}{t}}(\log t)\Big)^{m}\Big\{\mathbb{E}|Z[0,t]\cap Z'[0,t]|^{m}\Big\}^{1/2}\Big\}^{1/2}\\ &\times\Big\{\sum_{m=0}^\infty \frac{(\delta^{-1}\theta)^{m}}{m!}\Big(\sqrt{\frac{b_t}{t}}(\log t)\Big)^{m}\Big\{\mathbb{E}\zeta^{m}(t,\varepsilon)\Big\}^{1/2}\Big\}^{1/2},\end{aligned}$$

where in the last step we used (9.36) and the fact that $|Z[0,t]\cap Z'[0,t]|\ge 0$.

By [8, (2.11)], there is a $C>0$ independent of δ and θ such that

$$(9.38)\qquad \lim_{t\to\infty}\frac{1}{b_t}\log\sum_{m=0}^\infty \frac{(\delta\theta)^m}{m!}\Big(\sqrt{\frac{b_t}{t}}(\log t)\Big)^m\Big\{\mathbb{E}|Z[0,t]\cap Z'[0,t]|^m\Big\}^{1/2} = C(\delta\theta)^2.$$

In addition, by Lemma 9.1

$$(9.39)\qquad \sum_{m=0}^\infty \frac{(\delta^{-1}\theta)^m}{m!}\Big(\sqrt{\tfrac{b_t}{t}}(\log t)\Big)^m\Big\{\mathbb{E}\zeta^m(t,\varepsilon)\Big\}^{1/2}$$
$$\le \bigg\{\sum_{m=0}^\infty \frac{(\delta^{-1}\theta)^m}{m!}\Big(\sqrt{\tfrac{b_t}{t}}(\log t)\Big)^m\Big\{\mathbb{E}|\beta(t,\varepsilon)|^m\Big\}^{1/2}\bigg\}^{[b_t]},$$

where

$$(9.40)\quad \beta(t,\varepsilon) = \big|Z[0,\gamma_t]\cap Z'[0,\gamma_t]\big| - \sum_{x\in\mathbb{Z}^2} q_{t,\varepsilon}(x)\big|Z[0,\gamma_t]\cap\big(x+Z'[0,\gamma_t]\big)\big|.$$

Recall that $q_{t,\varepsilon}(x)$ is defined by (9.22) and $\gamma_t = t/[b_t]$. As in the proof of (9.23) we can check that

$$\sum_{x\in\mathbb{Z}^2} q_{t,\varepsilon}(x)\big|Z[0,\gamma_t]\cap\big(x+Z'[0,\gamma_t]\big)\big| = B_{\gamma_t,1},$$

see (8.2). By Lemma 8.1 (with t replaced by γ_t),

$$(9.41)\qquad \frac{b_t(\log t)^2}{t}\beta(t,\varepsilon) \xrightarrow{d} (2\pi)^2\det(\Gamma)\Big[\alpha([0,1]^2)-\alpha_\varepsilon([0,1]^2)\Big].$$

By Lemma 7.1 (with $p=2$),

$$(9.42)\quad \mathbb{E}|\beta(t,\varepsilon)|^m \le 2\sup_x \mathbb{E}^{(0,x)}|Z[0,\gamma_t]\cap Z'[0,\gamma_t]|^m \le m!C^m\Big(\frac{t}{b_t}(\log t)^{-2}\Big)^m.$$

Hence,

$$(9.43)\qquad \lim_{t\to\infty}\sum_{m=0}^\infty \frac{(\delta^{-1}\theta)^m}{m!}\Big(\sqrt{\tfrac{b_t}{t}}(\log t)\Big)^m\Big\{\mathbb{E}|\beta(t,\varepsilon)|^m\Big\}^{1/2}$$
$$= \sum_{m=0}^\infty \frac{(\delta^{-1}\theta)^m}{m!}\Big((2\pi)\sqrt{\det(\Gamma)}\Big)^m\Big\{\mathbb{E}\Big|\alpha([0,1]^2)-\alpha_\varepsilon([0,1]^2)\Big|^m\Big\}^{1/2}.$$

So by (9.39) we have

$$(9.44)\quad \limsup_{t\to\infty}\frac{1}{b_t}\log\sum_{m=0}^\infty \frac{(\delta^{-1}\theta)^m}{m!}\Big(\sqrt{\tfrac{b_t}{t}}(\log t)\Big)^m\Big\{\mathbb{E}\zeta^m(t,\varepsilon)\Big\}^{1/2}$$
$$\le \log\sum_{m=0}^\infty \frac{(\delta^{-1}\theta)^m}{m!}\Big((2\pi)\sqrt{\det(\Gamma)}\Big)^m\Big\{\mathbb{E}\Big|\alpha([0,1]^2)-\alpha_\varepsilon([0,1]^2)\Big|^m\Big\}^{1/2}.$$

By [25, Theorem 1, p.183],

$$\mathbb{E}\Big|\alpha([0,1]^2) - \alpha_\varepsilon([0,1]^2)\Big|^m \longrightarrow 0 \quad \text{as } \varepsilon \to 0^+, \tag{9.45}$$

for all $m \geq 1$. In addition, by [7, (1.12)], there is a constant $C > 0$ such that

$$\mathbb{E}\Big|\alpha([0,1]^2) - \alpha_\varepsilon([0,1]^2)\Big|^m \leq \mathbb{E}\alpha^m([0,1]^2) \leq m! C^m \tag{9.46}$$

for all $m \geq 1$. By dominated convergence, therefore,

$$\sum_{m=0}^{\infty} \frac{(\delta^{-1}\theta)^m}{m!} \Big((2\pi)\sqrt{\det(\Gamma)}\Big)^m \Big\{\mathbb{E}\Big|\alpha([0,1]^2) - \alpha_\varepsilon([0,1]^2)\Big|^m\Big\}^{1/2} \longrightarrow 1 \tag{9.47}$$

as $\varepsilon \to 0^+$. (Alternatively, this follows immediately from [10, (6.29)]). Thus

$$\lim_{\varepsilon\to 0^+} \limsup_{t\to\infty} \frac{1}{b_t} \log \sum_{m=0}^{\infty} \frac{(\delta^{-1}\theta)^m}{m!} \Big(\sqrt{\frac{b_t}{t}}(\log t)\Big)^m \Big\{\mathbb{E}\zeta^m(t,\varepsilon)\Big\}^{1/2} = 0. \tag{9.48}$$

Summarizing what we have,

$$\limsup_{\varepsilon\to 0^+} \limsup_{t\to\infty} \frac{1}{b_t} \log \mathbb{E} \cosh\Big\{\theta\sqrt{\frac{b_t}{t}}(\log t)|\xi(t,\varepsilon)|^{1/2}\Big\} \leq C(\delta\theta)^2. \tag{9.49}$$

Letting $\delta \to 0^+$ gives

$$\limsup_{\varepsilon\to 0^+} \limsup_{t\to\infty} \frac{1}{b_t} \log \mathbb{E} \cosh\Big\{\theta\sqrt{\frac{b_t}{t}}(\log t)|\xi(t,\varepsilon)|^{1/2}\Big\} = 0. \tag{9.50}$$

Since $\exp(x) \leq 2\cosh(x)$ we see from (9.50) that

$$\limsup_{\varepsilon\to 0^+} \limsup_{t\to\infty} \frac{1}{b_t} \log \mathbb{E} \exp\Big\{\theta\sqrt{\frac{b_t}{t}}(\log t)|\xi(t,\varepsilon)|^{1/2}\Big\} = 0. \tag{9.51}$$

By (9.28) and (9.23) we have thus completed the proof of Lemma 7.8 when $j = 0$. If in (7.22) with $j = 0$ we replace t by $2^{-j}t$, θ by $2^{-j/2}\theta$, b_t by $\widetilde{b}_t =: b_{2^j t}$ and ε by $2^{j/2}\varepsilon$, we obtain (7.22) for any j (compare the proof of (7.20)). □

10 Laws of the iterated logarithm

We first prove some lemmas in preparation for the proof of Theorem 1.6. Define

$$\widetilde{\varphi}_j = \frac{j}{\mathcal{H}(j)}, \qquad \widetilde{G}_j = (R_j - \widetilde{\varphi}_j)\frac{(\log n)^2}{n},$$

and $K = [\log\log n] + 1$.

Lemma 10.1 *There exists a constant c_1 such that if A and B are positive integers and $C = A + B$, then*

$$|\widetilde{\varphi}_C - \widetilde{\varphi}_A - \widetilde{\varphi}_B| \le c_1 \frac{(A \wedge B)^{1/2}}{C^{1/2}(\log C)^2}.$$

Proof. The cases when A or B equal 1 are easy, so we suppose $A, B > 1$. Write

$$\widetilde{\varphi}_C - \widetilde{\varphi}_A - \widetilde{\varphi}_B = \frac{C}{\mathcal{H}(C)}\Big[-\frac{A}{C}\frac{\mathcal{H}(C)-\mathcal{H}(A)}{\mathcal{H}(A)} - \frac{B}{C}\frac{\mathcal{H}(C)-\mathcal{H}(B)}{\mathcal{H}(B)}.\Big]$$

By (4.23) and (4.24), the right hand side is bounded in absolute value by

$$c_2\frac{C}{\log C}\Big[-\frac{A}{C}\frac{\log C - \log A}{\log A} - \frac{B}{C}\frac{\log C - \log B}{\log B}\Big].$$

We now follow Lemma 4.2 of [5]. If $2 \le A \le C^{1/2}$, then $\log A \ge \frac{1}{3}$ and

$$0 \le \frac{A}{C}\frac{\log C - \log A}{\log A} \le 3\Big(\frac{A}{C}\Big)^{1/2}\frac{1}{\log C}\frac{(\log C)^2}{C^{1/4}} \le \frac{c_3}{\log C}\Big(\frac{A}{C}\Big)^{1/2}.$$

If $C^{1/2} \le A \le C/2$, then

$$0 \le \frac{A}{C}\frac{\log C - \log A}{\log A} \le 2\frac{A}{C}\frac{\log(C/A)}{\log C} \le \frac{c_4}{\log C}\Big(\frac{A}{C}\Big)^{1/2}.$$

If $A \ge C/2$, then

$$0 \le \frac{A}{C}\frac{\log C - \log A}{\log A} \le \frac{c_5}{\log A}|\log(1-(B/C))| \le \frac{c_4}{\log C}\Big(\frac{B}{C}\Big)^{1/2}.$$

We similarly bound $(B/C)((\log C - \log B)/\log B)$. $\square$

Lemma 10.2 *There exists λ_0 such that if $\lambda \geq \lambda_0$, then*

$$\mathbb{P}(\max_{m\leq n} \overline{R}_m > \lambda n \log\log\log n/(\log n)^2) \leq (\log n)^{-2}.$$

Proof. We first prove that there exists $M > 0$ not depending on n such that

$$\mathbb{P}\left(\max_{1\leq j\leq n} \widetilde{G}_j > M\right) \leq 1/2. \tag{10.1}$$

Let θ_j be the usual shift operators. Since $R_n - R_m \leq R_{n-m} \circ \theta_m$, then by Lemma 10.1

$$\widetilde{G}_n - \widetilde{G}_m \leq \widetilde{G}_{n-m} \circ \theta_m + c_1\Big(\frac{m}{n} \wedge \frac{n-m}{n}\Big)^{1/2}. \tag{10.2}$$

By the Markov property, (5.1) and Theorem 5.4,

$$\mathbb{E}[(\widetilde{G}_j \circ \theta_m)^2] = \mathbb{E}^{S_m}\widetilde{G}_j^2 = \mathbb{E}\widetilde{G}_j^2 \leq c_2(j/n)^2\Big(\frac{\log n}{\log j}\Big)^4 \leq c_3(j/n)^{3/2}. \tag{10.3}$$

In particular

$$\mathbb{E}\widetilde{G}_j^2 \leq c_3(j/n)^{3/2}. \tag{10.4}$$

For each k let k_j be the largest element of $\{[mn/2^j] : m \leq 2^j\}$ that is less than or equal to k. We have

$$\widetilde{G}_k = \widetilde{G}_{k_0} + (\widetilde{G}_{k_1} - \widetilde{G}_{k_0}) + (\widetilde{G}_{k_2} - \widetilde{G}_{k_1}) + \cdots,$$

where the sum is a finite sum. If $\max_{k\leq n} \widetilde{G}_k \geq M$, then for some $j \geq 0$

$$\widetilde{G}_{[(m+1)n/2^j]} - \widetilde{G}_{[mn/2^j]} > \frac{M}{40(j+1)^2} \qquad \text{for some } m \leq 2^j. \tag{10.5}$$

Let $I(m,j) = [(m+1)n/2^j] - [mn/2^j]$. If $m \leq 2^{j/8}$, then by (10.4)

$$\begin{aligned}
&\mathbb{P}\Big(\widetilde{G}_{[(m+1)n/2^j]} - \widetilde{G}_{[mn/2^j]} > \frac{M}{40(j+1)^2}\Big)\\
&\qquad \leq \frac{3200(j+1)^4}{M^2}(\mathbb{E}\widetilde{G}^2_{[(m+1)n/2^j]} + \mathbb{E}\widetilde{G}^2_{[mn/2^j]})\\
&\qquad \leq c_4(j+1)^4(m/2^j)^{3/2}/M^2\\
&\qquad \leq c_5/(2^{5j/4}M^2).
\end{aligned}$$

If $m > 2^{j/8}$, then using (10.2)

$$\begin{aligned}\widetilde{G}_{[(m+1)n/2^j]} - \widetilde{G}_{[mn/2^j]} &\leq \widetilde{G}_{I(m,j)} \circ \theta_{[mn/2^j]} + c_1(m+1)^{-1/2} \\ &\leq \widetilde{G}_{I(m,j)} \circ \theta_{[mn/2^j]} + \frac{M}{80(j+1)^2}\end{aligned}$$

if M is large enough. In this case, using (10.3),

$$\begin{aligned}&\mathbb{P}(\widetilde{G}_{[(m+1)n/2^j]} - \widetilde{G}_{[mn/2^j]} > \frac{M}{40(j+1)^2}) \\ \leq\ &\mathbb{P}(\widetilde{G}_{I(m,j)} \circ \theta_{[mn/2^j]} > \frac{M}{80(j+1)^2}) \\ \leq\ &c_6 \frac{(j+1)^4}{M^2} \frac{1}{2^{3j/2}} \\ \leq\ &c_7/(2^{5j/4} M^2).\end{aligned}$$

We thus have

$$\begin{aligned}\mathbb{P}(\max_{j\leq n} \widetilde{G}_j > M) &\leq \sum_{j=0}^{\infty} \sum_{m=1}^{2^j} \mathbb{P}(\widetilde{G}_{[(m+1)n/2^j]} - \widetilde{G}_{[mn/2^j]} > \frac{M}{40(j+1)^2}) \\ &\leq \sum_{j=0}^{\infty} c_8 \frac{2^j}{M^2} \frac{1}{2^{5j/4}} \\ &\leq \frac{c_8}{M^2} \leq \frac{1}{2}\end{aligned}$$

if M is large enough.

We next prove there exists c_9 and c_{10} such that

$$\mathbb{E}\left[\exp\left(c_9 \max_{1\leq j\leq n} \widetilde{G}_j\right)\right] \leq c_{10}. \tag{10.6}$$

Note that by (10.2), we have

$$\widetilde{G}_n - \widetilde{G}_m \leq \widetilde{G}_{n-m} \circ \theta_m + c_{11}. \tag{10.7}$$

Now, choose c_{12} large so that $c_{12}/2 > c_{11}$ and

$$\mathbb{P}(\max_{1\leq j\leq n} \widetilde{G}_j > (c_{12}/2) - c_{11}) < 1/2 \qquad \text{for all } n \geq 1, \tag{10.8}$$

which is possible by (10.1). Let $T_k = \min\{j : \widetilde{G}_j > c_{12}k\}$. Then

$$\begin{aligned}\mathbb{P}(\max_{j\le n}\widetilde{G}_j > c_{12}(k+1)) &= \mathbb{P}(T_{k+1}\le n)\\ &\le \mathbb{P}(T_k\le n, \max_{T_k\le j\le n}(\widetilde{G}_j-\widetilde{G}_{T_k}) > c_{12}/2)\\ &= \mathbb{E}[\mathbb{P}(\max_{T_k\le j\le n}(\widetilde{G}_j-\widetilde{G}_{T_k}) > c_{12}/2|\mathcal{F}_{T_k}); T_k\le n]\\ &\le \mathbb{E}[\mathbb{P}(\max_{j\le n}\widetilde{G}_j > (c_{12}/2)-c_{11}); T_k\le n]\\ &\le \frac{1}{2}\mathbb{P}(T_k\le n),\end{aligned}$$

where the second inequality follows by (10.7) and the third inequality by (10.8). By induction we obtain $\mathbb{P}(T_k\le n)\le 2^{-n}$, which yields (10.6).

Let

$$C_j = \max_{[jn/K]\le i<[(j+1)n/K]}\quad [R_i - R_{[jn/K]} - \zeta\widetilde{\varphi}_{i-[jn/K]}]$$

and

$$D_j = \frac{C_j}{(n/K)/(\log(n/K))^2},$$

where $\zeta = 2\pi\sqrt{\det\Gamma}$. By (10.6) there exist c_{13}, c_{14} such that $\mathbb{E}e^{c_{13}D_j}\le c_{14}$. Moreover, the D_j are independent. Let

$$e_{K,n} = |\widetilde{\varphi}_n - K\widetilde{\varphi}_{[n/K]}|/(n/(\log n)^2).$$

An elementary computation shows that

$$e_{K,n}\le c_{15}\log K.$$

Since

$$\max_{m\le n}\frac{(R_m-\zeta\widetilde{\varphi}_m)}{n/(\log n)^2}\le\frac{c_{16}}{K}\sum_{j=1}^{K}D_j+\zeta e_{K,n}$$

for $A \geq 2c_{15}\zeta$, we have

$$
\begin{aligned}
\mathbb{P}(\max_{m\leq n} \frac{R_m - \zeta\widetilde{\varphi}_m}{n/(\log n)^2} > A\log K) &\leq \mathbb{P}(\frac{c_{16}}{K}\sum_{j=1}^{K} D_j > A\log K - \zeta e_{K,n}) \\
&\leq \mathbb{P}(\sum_{j=1}^{K} D_j > AK(\log K)/(2c_{16})) \\
&\leq e^{-c_{15}AK(\log K)/(2c_{16})}\mathbb{E}e^{c_{15}\sum D_j} \\
&\leq e^{-c_{17}AK(\log K)/2}c_{17}^K \\
&\leq e^{-c_{17}AK(\log K)/2}
\end{aligned}
$$

if K is large enough.

So

$$\mathbb{P}(\max_{m\leq n} \widetilde{G}_m > A\log\log\log n) \leq (\log n)^{-2}$$

if A is large enough. By (5.1) and (4.23), we see that

$$\max_{m\leq n} |\overline{R}_m - (R_m - \widetilde{\varphi}_m)| \leq c_{18}\frac{n}{(\log n)^2} = o(n\log\log\log n/(\log n)^2),$$

and our result now follows immediately. □

Proof of Theorem 1.6: Let $\xi = 2\pi\sqrt{\det\Gamma}$. We begin with the upper bound. Let $\eta, \varepsilon > 0$ be small and let $q > 1$ be very close to 1. Let $t_i = [q^i]$. If

$$A_i = \left\{\overline{R}_{t_i} \geq (1+\eta)\xi t_i \log\log\log t_i/(\log t_i)^2\right\},$$

then it follows from Theorem 1.1 that $\sum_i \mathbb{P}(A_i) < \infty$, and so by Borel-Cantelli, $\mathbb{P}(A_i \text{ i.o.}) = 0$.

Next, if λ is sufficiently large,

$$\mathbb{P}(\max_{m\leq n} \overline{R}_m > \lambda n\log\log\log n/(\log n)^2) \leq (\log n)^{-2}; \tag{10.9}$$

by Lemma 10.2. Let

$$B_i = \left\{\max_{t_i\leq k\leq t_{i+1}} [\overline{R}_k - \overline{R}_{t_i}] > \varepsilon t_i \log\log\log t_i/\log^2 t_i\right\}.$$

By subadditivity $R_k - R_{t_i} \le R_{k-t_i} \circ \theta_{t_i}$, where θ_{t_i} is the usual shift operator of Markov theory. By Lemma 10.1

$$\mathbb{E}R_k - \mathbb{E}R_{t_i} \ge \mathbb{E}R_{k-t_i} - c\frac{t_i}{\log^2 t_i}. \tag{10.10}$$

So by the Markov property, and using the fact that the $\mathbb{P}^x$ law of R_{k-t_i} does not depend on x, for i large

$$\begin{aligned}
(10.11)\quad & \mathbb{P}(B_i) \\
&= \mathbb{P}(\max_{t_i \le k \le t_{i+1}} [R_k - R_{t_i} - (\mathbb{E}R_k - \mathbb{E}R_{t_i})] > \varepsilon t_i \log\log\log t_i / \log^2 t_i) \\
&\le \mathbb{P}(\max_{t_i \le k \le t_{i+1}} [R_k - R_{t_i} - \mathbb{E}R_{k-t_i}] + c\frac{t_i}{\log^2 t_i} > \varepsilon t_i \log\log\log t_i / \log^2 t_i) \\
&\le \mathbb{P}^{S_{t_i}}(\max_{t_i \le k \le t_{i+1}} [\overline{R}_{k-t_i}] > \varepsilon t_i \log\log\log t_i / \log^2 t_i - c\frac{t_i}{\log^2 t_i}) \\
&\le \mathbb{P}(\max_{k \le t_{i+1} - t_i} \overline{R}_k \ge \tfrac{\varepsilon}{2} t_i \log\log\log t_i / (\log t_i)^2).
\end{aligned}$$

If q is sufficiently small, then $\sum_i \mathbb{P}(B_i)$ will be summable by (10.9). So with probability one, for i large enough

$$\max_{k \le t_{i+1}} \overline{R}_k \le ((1+\eta)\xi + \varepsilon) q t_i \log\log\log t_i / (\log t_i)^2.$$

Since η and ε are arbitrary, and we can take q as close to 1 as we like, this implies the upper bound.

Let $\eta > 0$, $t_i = [\exp(i^{1+\frac{\eta}{2}})]$, $V_i = \#S((t_i, t_{i+1}])$, and set

$$C_i = \left\{ \overline{V}_i > (1-\eta)\xi(t_{i+1} - t_i) \log\log\log(t_{i+1} - t_i) / (\log(t_{i+1} - t_i))^2 \right\}.$$

Note that the events C_i are independent. By Theorem 1.1 and Borel-Cantelli, $\mathbb{P}(C_i \text{ i.o.}) = 1$. Note

$$\frac{(t_{i+1} - t_i) \log\log\log(t_{i+1} - t_i)}{(\log(t_{i+1} - t_i))^2} = \frac{t_{i+1} \log\log\log t_{i+1}}{(\log t_{i+1})^2} \Big(1 + o(1)\Big).$$

Also

$$|V_i - R_{t_{i+1}}| + |\mathbb{E}V_i - \mathbb{E}R_{t_{i+1}}| \le 2t_i = o\Big(\frac{t_{i+1} \log\log\log t_{i+1}}{(\log t_{i+1})^2}\Big).$$

Therefore with probability one, infinitely often

$$\overline{R}_{t_{i+1}} > \Big(1-\frac{\eta}{2}\Big)\xi t_{i+1} \log\log\log t_{i+1}/(\log t_{i+1}).$$

This proves the lower bound. □

We now turn to the LIL for $-\overline{R}_n$. First we prove

Lemma 10.3 *Let $\varepsilon > 0$. There exists $q_0(\varepsilon)$ such that if $1 < q < q_0(\varepsilon)$, then*

$$\mathbb{P}(\max_{[q^{-1}n]\le k\le n}(\overline{R}_n - \overline{R}_k) > \varepsilon n \log\log n/(\log n)^2) \le \frac{1}{(\log n)^2}$$

for n large.

Proof. Let

$$G_k = (R_n - R_k)\frac{(\log n)^2}{n}.$$

Let

$$\mathcal{A}_i = \Big\{[q^{-1}n] + \Big[\frac{n\ell}{2^i}\Big] : \ell \in \mathbb{Z}_+\Big\} \cap [0,n], \qquad i \le \log\log n + 1.$$

Given k, let $k_i = \max\{j \in \mathcal{A}_i : j \le k\}$. We write

$$G_k = G_{k_1} + (G_{k_2} - G_{k_1}) + (G_{k_3} - G_{k_2}) + \cdots,$$

where the sum is actually a finite one. If $\overline{G}_k > \varepsilon \log\log n$ for some $[q^{-1}n] \le k \le n$, then either

(10.12) $$\overline{G}_{[q^{-1}n]} > \tfrac{\varepsilon}{2}\log\log n$$

or for some i there exist consecutive elements ℓ, m of $\mathcal{A}_i$ such that

(10.13) $$\overline{G}_m - \overline{G}_\ell > \frac{\varepsilon}{10i^2}\log\log n.$$

By subadditivity $R_n - R_k \le R_{n-k}\circ\theta_k$ for $k \le n$, while by Lemma 10.1

$$\mathbb{E}R_n - \mathbb{E}R_k \ge \mathbb{E}R_{n-k} - c_1(1-q^{-1})^{1/2}\frac{n}{(\log n)^2}.$$

Then setting $k = [q^{-1}n]$,

$$\begin{aligned}
&\mathbb{P}(\overline{G}_{[q^{-1}n]} > \tfrac{\varepsilon}{2}\log\log n)\\
&\quad = \mathbb{P}\Big((R_n - R_k)\frac{(\log n)^2}{n} - (\mathbb{E}R_n - \mathbb{E}R_k)\frac{(\log n)^2}{n} > \tfrac{\varepsilon}{2}\log\log n\Big)\\
&\quad \le \mathbb{P}^{S_k}\Big(R_{n-k}\frac{(\log n)^2}{n} - \mathbb{E}R_{n-k}\frac{(\log n)^2}{n} + c_1(1-q^{-1})^{1/2} > \tfrac{\varepsilon}{2}\log\log n\Big).
\end{aligned}$$

Using the fact that the $\mathbb{P}^x$ law of R_{n-k} does not depend on x, this is the same as

$$\mathbb{P}\Big(\frac{\overline{R}_{n-k}}{(n-k)/(\log(n-k))^2} > \frac{n}{n-k}\frac{(\log(n-k))}{(\log n)^2}\Big(\tfrac{\varepsilon}{2}\log\log n - c_1(1-q^{-1})^{1/2}\Big)\Big).$$

If $q > 1$ is close enough to 1 and n is large enough, by Theorem 1.5 this is bounded by

$$c_2\exp\Big(-c_3\frac{\varepsilon}{2}\frac{1}{1-q^{-1}}\log\log n\Big) \le \frac{1}{2(\log n)^2}. \tag{10.14}$$

This bounds the probability of the event described in (10.12).

Similarly, $R_m - R_\ell \le R_{m-\ell}\circ\theta_\ell$ and by Lemma 10.1

$$\mathbb{E}R_m - \mathbb{E}R_\ell \ge \mathbb{E}R_{m-\ell} - c_1\Big(\frac{m-\ell}{n}\Big)^{1/2}\frac{n}{(\log n)^2}.$$

So if ℓ and m are consecutive elements of $\mathcal{A}_i$, similarly to (10.14) we obtain

$$\mathbb{P}\Big(\overline{G}_m - \overline{G}_\ell \ge \frac{\varepsilon}{10i^2}\log\log n\Big) \le \mathbb{P}\Big(\frac{\overline{R}_{m-\ell}}{n/(\log n)^2} \ge \frac{\varepsilon}{10i^2}\log\log n - c_1 2^{-i/2}\Big). \tag{10.15}$$

For n large, $c_1 2^{-i/2} \le \frac{\varepsilon}{20i^2}\log\log n$ for all i and $n/(m-\ell) = 2^i$, so by Theorem 1.5 the left hand side of (10.15) is less than

$$\mathbb{P}\Big(\frac{\overline{R}_{m-\ell}}{(m-\ell)/(\log(m-\ell))^2} \ge \frac{\varepsilon}{40i^2}\frac{n}{m-\ell}\log\log n\Big) \le c_2\exp\Big(-c_3\frac{\log\log n}{40i^2}2^i\Big).$$

There are at most 2^{i+1} such pairs ℓ, m, so

$$\begin{aligned}
w_i &:= \mathbb{P}(\text{for some consecutive elements } \ell, m \in \mathcal{A}_i : \overline{G}_m - \overline{G}_\ell > \frac{\varepsilon}{10i^2}\log\log n)\\
&\le c_2 2^{i+1}\exp\Big(-c_3\frac{\log\log n}{40i^2}2^i\Big).
\end{aligned}$$

Since $c_3 2^i/40i^2 > 2(i+1)\log 2$ for i large, then for n large enough

$$w_i \le c_2 \exp\Big(- c_3 \frac{2^i \log\log n}{40 i^2}\Big).$$

So then

$$\sum_{i=1}^{\infty} w_i \le \frac{1}{2(\log n)^2}$$

for large n, and this bounds the event that for some i there exist consecutive elements ℓ, m of $\mathcal{A}_i$ such that (10.13) holds. Combining with the bound for (10.12), the result follows. □

Proof of Theorem 1.7: Let

$$\Theta = (2\pi)^2 \det(\Gamma)^{-1/2} \kappa(2,2)^{-4}.$$

Upper bound. Let $\eta, \varepsilon > 0$ and choose $q \in (1, q_0(\varepsilon))$ where $q_0(\varepsilon)$ is as in Lemma 10.3. Let $t_i = [q^i]$. If

$$A_i = \left\{ -\overline{R}_{t_i} > (1+\eta)\Theta^{-1} \frac{t_i \log\log t_i}{(\log t_i)^2} \right\},$$

then by Theorem 1.5, $\sum_i \mathbb{P}(A_i) < \infty$, and hence by Borel-Cantelli, $\mathbb{P}(A_i \text{ i.o.}) = 0$. Let

$$B_i = \left\{ \max_{t_i \le k \le t_{i+1}} (\overline{R}_{t_{i+1}} - \overline{R}_k) > \varepsilon \frac{t_{i+1} \log\log t_{i+1}}{(\log t_{i+1})^2} \right\}.$$

By Lemma 10.3, $\sum_i \mathbb{P}(B_i) < \infty$, and again $\mathbb{P}(B_i \text{ i.o.}) = 0$. So with probability one, for k large we have $t_i \le k \le t_{i+1}$ for some i large, and then

$$\begin{aligned} -\overline{R}_k &= -\overline{R}_{t_{i+1}} + (\overline{R}_{t_{i+1}} - \overline{R}_k) \\ &\le \Theta^{-1}(1+\eta) \frac{t_{i+1} \log\log t_{i+1}}{(\log t_{i+1})^2} + \varepsilon \frac{t_{i+1} \log\log t_{i+1}}{(\log t_{i+1})^2} \\ &\le q(\Theta^{-1}(1+2\eta) + 2\varepsilon) \frac{k \log\log k}{(\log k)^2}. \end{aligned}$$

Since ε, η can be made as small as we like and we can take q as close to 1 as we like, this gives the upper bound.

Lower bound. Let $\eta > 0$, $t_i = [\exp(i^{1+\frac{\eta}{2}})]$, $V_i = \#S((t_i, t_{i+1}])$. Let

$$C_i = \left\{ -\overline{V}_i \geq \Theta^{-1}(1-\eta)\frac{(t_{i+1}-t_i)\log\log(t_{i+1}-t_i)}{(\log(t_{i+1}-t_i))^2} \right\}.$$

By Theorem 1.5, $\sum_i \mathbb{P}(C_i) = \infty$. The C_i are independent, and so by Borel-Cantelli, $\mathbb{P}(C_i \text{ i.o.}) = 1$.

Since $R_{t_{i+1}} \leq V_i + R_{t_i}$ and $\mathbb{E}R_{t_{i+1}} \geq \mathbb{E}V_i$, then

$$-\overline{R}_{t_{i+1}} \geq -\overline{V}_i - R_{t_i}.$$

Now

$$R_{t_i} \leq t_i = o\Big(\frac{t_{i+1}\log\log t_{i+1}}{(\log t_{i+1})^2}\Big)$$

and

$$\frac{(t_{i+1}-t_i)\log\log(t_{i+1}-t_i)}{(\log(t_{i+1}-t_i))^2} \sim \frac{t_{i+1}\log\log t_{i+1}}{(\log t_{i+1})^2},$$

so

$$-\overline{R}_{t_{i+1}} \geq \Theta^{-1}(1-2\eta)\frac{t_{i+1}\log\log t_{i+1}}{(\log t_{i+1})^2}, \qquad i.o.$$

This implies the lower bound. □

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