Regularity of Harmonic Functions for a Class of Singular Stable-like Processes

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Abstract

We consider the system of stochastic differential equations

$$dX_t = A(X_{t-})\, dZ_t,$$

where $Z_t^1, \ldots, Z_t^d$ are independent one-dimensional symmetric stable processes of order α, and the matrix-valued function A is bounded, continuous and everywhere non-degenerate. We show that bounded harmonic functions associated with X are Hölder continuous, but a Harnack inequality need not hold. The Lévy measure associated with the vector-valued process Z is highly singular.

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1 Introduction

A one-dimensional symmetric stable process of index $\alpha \in (0, 2)$ is the Lévy process taking values in $\mathbb{R}$ with no drift, no Gaussian part, and Lévy measure

$$n(dh) = c_1/|h|^{1+\alpha}\, dh.$$

Let $Z_t = (Z_t^1, \ldots, Z_t^d)$ be a vector of d independent one-dimensional symmetric stable processes of index α. Consider the system of stochastic differential equations

$$dX_t^i = \sum_{j=1}^{d} A_{ij}(X_{t-})\, dZ_t^j, \qquad X_0^i = x_0^i, \qquad i = 1, \ldots, d, \tag{1.1}$$

where $x_0 = (x_0^1, \ldots, x_0^d) \in \mathbb{R}^d$ and $A(x)$ is a bounded $d \times d$ matrix-valued function that is continuous in x and everywhere non-degenerate, that is, the determinant $\det(A(x)) \neq 0$ for all x. The main

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result of [2] is that under these conditions there is a unique weak solution to the system (1.1) and the family $\{X, \mathbb{P}^{x_0}, x_0 \in \mathbb{R}^d\}$ forms a strong Markov process on $\mathbb{R}^d$. The process X may be referred to as stable-like because it possesses an approximate scaling property similar to the stable processes; see [4] and [5] for other examples where the term stable-like has been used. The system (1.1) has been suggested as a possible model for a financial market with jumps in the security prices ([6]). Note that by Proposition 4.1 of [2], the infinitesimal generator of the Markov process X determined by (1.1) is

$$\mathcal{L}f(x) = \sum_{j=1}^d \int_{\mathbb{R}\setminus\{0\}} \left(f(x + a_j(x)w) - f(x) - w1_{\{|w|\le 1\}} \nabla f(x) \cdot a_j(x) \right) \frac{c_1}{|w|^{1+\alpha}} dw, \qquad (1.2)$$

where $a_j(x)$ is the j^{th} column of the matrix $A(x)$. Associated with the operator $\mathcal{L}$ is the symbol

$$\ell(x,u) := c_2 \sum_{j=1}^d |u \cdot a_j(x)|^\alpha, \qquad x, u \in \mathbb{R}^d.$$

This means

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} \ell(x,u) e^{-iu\cdot x} \widehat{f}(u)\, du,$$

where $\widehat{f}$ denotes the Fourier transform of f. This is an example of a pseudodifferential operator with singular state-dependent symbol.

We say that a function h that is bounded in $\mathbb{R}^d$ is harmonic (with respect to X) in a domain D if $h(X_{t\wedge\tau_D})$ is a martingale with respect to $\mathbb{P}^x$ for every $x \in D$, where τ_D is the time of first exit from D. The process X is shown to have no explosions in finite time in [2] and when D is bounded, it is easy to see from (1.1) that $\mathbb{P}^x(\tau_D < \infty) = 1$ for every $x \in D$. So by the bounded convergence theorem and the strong Markov property of X, a bounded function h on $\mathbb{R}^d$ is harmonic in a bounded domain D if and only if

$$h(x) = \mathbb{E}^x[h(X_{\tau_D})] \qquad \text{for every } x \in D.$$

Consequently, every bounded harmonic function in a bounded domain D is the difference of two non-negative bounded harmonic functions in D. It follows from Proposition 4.1 of [2] that a bounded C^2 function u is harmonic in D if and only if $\mathcal{L}u = 0$ in D.

The main goal of this paper is to prove the Hölder continuity of functions which are bounded and harmonic with respect to X in a domain.

There are two reasons why the Hölder continuity is perhaps a bit unexpected. Consider the case where A is identically equal to the identity matrix, and so $X \equiv Z$. Even in this case a Harnack inequality may fail; see Section 3. Nevertheless the Hölder continuity of the harmonic functions holds. The other reason is that the process Z is quite singular. It is a Lévy process, but the support of its Lévy measure is the union of the coordinate axes. By contrast, the support of the Lévy measure for a d-dimensional (rotationally) symmetric stable process is all of $\mathbb{R}^d$, a much more tractable situation.

The key to our method is the technique of Krylov-Safonov as given, for example, in the exposition in [1]. The most difficult step in our proof is the proof of a support theorem for X; this is given in Section 2. We remark that the current paper is the first one where the full strength of the Krylov-Safonov technique has been used in the context of pure jump processes.

For a Borel subset $C \subset \mathbb{R}^d$, let $T_C := \inf\{t \ge 0 : X_t \in C\}$ and $\tau_C := \inf\{t \ge 0 : X_t \notin C\}$ be the first entrance and departure time of C by X. Let $|C|$ denote the Lebesgue measure of a Borel set

C. The open ball of radius r centered at x will be denoted as $B(x,r)$. The paths of Z_t are right continuous with left limits. We write

$$Z_{t-} := \lim_{s\uparrow t, s<t} Z_s, \qquad \Delta Z_t := Z_t - Z_{t-},$$

and similarly X_{t-} and ΔX_t. The letter c with a subscript denotes a positive finite constant whose exact value is unimportant and may vary from one usage to the next. Constant c typically depends on α and d, but for convenience this dependence will not be explicitly mentioned throughout the paper.

2 Regularity

For $1 \le i \le d$, denote by e_i the unit vector in the x_i direction in $\mathbb{R}^d$. Let $x_0 \in \mathbb{R}^d$ and let $B = B(x_0, 1)$. For simplicity, we write τ for τ_B. We will use $A(x)^{-1}$ to denote the inverse matrix of $A(x)$.

Proposition 2.1 *There exist positive constants c_1, c_2 that depend only on the upper bound of $A(x)$ and $A(x)^{-1}$ on B such that*
(a) $\mathbb{E}^x[\tau] \le c_1$ for all $x \in B$;
(b) $\mathbb{E}^x[\tau] \ge c_2$ for all $x \in B(x_0, \frac{1}{2})$.

Proof. (a) Let $A_0 = \inf\{|A(x)(e_1)| : x \in \overline{B}\}$. We know $A_0 > 0$ because $A(x)$ is continuous in x and nondegenerate for each x. Since the Z^i's are independent one-dimensional symmetric α-stable process, no two of them make a jump at the same time. So there exists a positive constant c_3 such that

$$\mathbb{P}\left(\exists s \le 1 : \Delta Z^1_s > 3/A_0 \text{ but } \Delta Z^k_s = 0 \text{ for } k = 2, \cdots, d\right) \ge c_3.$$

Suppose there exists $s \in [0,1]$ such that $\Delta Z^1_s > 3/A_0$, $\Delta Z^k_s = 0$ for $k = 2, \cdots, d$, and $X_{s-} \in B$. Then by (1.1)

$$|\Delta X^1_s| = |\Delta Z^1_s|\,|A(X_{s-})e_1| > 3$$

if $X_{s-} \in \overline{B}$. So with probability at least c_3, X will have left B by time 1. Hence if $x \in B$,

$$\mathbb{P}^x(\tau > 1) \le 1 - c_3.$$

Let $\{\theta_t, t > 0\}$ denotes the usual shift operators for X. By the Markov property,

$$\begin{aligned}\mathbb{P}^x(\tau > m+1) &\le \mathbb{P}^x(\tau > m, \tau \circ \theta_m > 1)\\ &= \mathbb{E}^x[\mathbb{P}^{X_m}(\tau > 1); \tau > m]\\ &\le (1 - c_3)\mathbb{P}^x(\tau > m).\end{aligned}$$

By induction,

$$\mathbb{P}^x(\tau > m) \le (1 - c_3)^m,$$

and (a) follows.

(b) Let

$$\widetilde{Z}^i_t := \sum_{s \le t} \Delta Z^i_s 1_{(|\Delta Z^i_s| > 1)} \qquad \text{and} \qquad \overline{Z}^i_t := Z^i_t - \widetilde{Z}^i_t.$$

Note

$$\mathbb{E}\,[\overline{Z}^i, \overline{Z}^i]_t = t \int_{-\beta}^{\beta} x^2 \frac{c_4}{|x|^{1+\alpha}} dx = c_5 t \beta^{2-\alpha}.$$

Let $\overline{X}$ solve

$$d\overline{X}_t = A(\overline{X}_t)\, d\overline{Z}_t.$$

Note that for each $i = 1, \cdots, d$, $\overline{X}^i$ is a purely discontinuous square integrable martingale with $|\Delta \overline{X}^i_t| \leq c_6 \sum_{j=1}^d |\Delta \overline{Z}^j_t|$. Hence

$$[\overline{X}^i, \overline{X}^i]_t \leq c_7 \sum_{j=1}^d [\overline{Z}^j, \overline{Z}^j]_t.$$

First by Chebyshev's inequality and then by Doob's inequality,

$$\begin{aligned}
\mathbb{P}^x\Big(\sup_{s\leq t} |\overline{X}^i_s - \overline{X}^i_0| > \frac{1}{4d}\Big) &\leq 16d^2\, \mathbb{E}\, \Big[\sup_{s\leq t} |\overline{X}^i_s - \overline{X}^i_0|^2\Big] \\
&\leq 64d^2\, \mathbb{E}\, \Big[(\overline{X}^i_t - \overline{X}^i_0)^2\Big] \\
&= 64d^2\, \mathbb{E}\, [\overline{X}^i, \overline{X}^i]_t \\
&\leq c_8 \sum_{j=1}^d \mathbb{E}\, [\overline{Z}^j, \overline{Z}^j]_t \\
&\leq c_9 t.
\end{aligned}$$

Choose t small so that $c_9 t \leq 1/4$.

We can choose t smaller if necessary so that

$$\mathbb{P}(\widetilde{Z}^j_s \neq 0 \text{ for some } s \in [0, t]) \leq 1/(4d).$$

So there exists t such that $\mathbb{P}(\overline{Z}_s \neq Z_s$ for some $s \in [0, t]) \leq 1/4$, and it follows that

$$\mathbb{P}(\overline{X}_s \neq X_s \text{ for some } s \in [0, t]) \leq 1/4.$$

Therefore with probability at least $1/2$ we have $\sup_{s\leq t} |X_s - X_0| \leq 1/4$ and so in particular

$$\mathbb{P}^x(\tau > t) \geq 1/2 \qquad \text{for } x \in B(x_0, \tfrac{1}{2}).$$

Consequently, we have $\mathbb{E}\,^x \tau \geq t \mathbb{P}^x(\tau \geq t) \geq t/2$ for $x \in B(x, \frac{1}{2})$. □

Proposition 2.2 *There exist constants $\eta_0 > 0, p_0 \geq 2$, and c_1 that depend only on the upper bound of $A(x)$ and $A(x)^{-1}$ on B such that if the oscillation of A on $B(x_0, 1)$ is less than η_0, then*

$$\mathbb{E}\,^x \left[\int_0^\tau 1_C(X_s)\, ds\right] \leq c_1 |C|^{1/p_0}, \qquad x \in B.$$

Proof. Note that the process $\{X_t, t \leq \tau\}$ is determined by the matrix A on B only. Without loss of generality, for this proof we redefine A for $x \notin B$ so that A is continuous on $\mathbb{R}^d$ and

$$\eta := \sup_{x\in\mathbb{R}^d} \|A(x) - A(x_0)\| = \sup_{x\in B} \|A(x) - A(x_0)\|.$$

Let R_λ and $\mathcal{L}_0$ be the resolvent and infinitesimal generator of the Levy process $Y_t = Y_0 + A(x_0)Z_t$, $\mathcal{L}$ the infinitesimal generator of X, S_λ the resolvent of X, and $\mathcal{B} := \mathcal{L} - \mathcal{L}_0$. There exist $\eta_0 > 0$ and $p_0 \geq 2$ so that the conclusion of Proposition 5.2 of [2] holds, namely, $\|\mathcal{B}R_\lambda f\|_{p_0} \leq \frac{1}{4}\|f\|_{p_0}$. For $f \in L^{p_0}(\mathbb{R}^d)$, set $h = f - \lambda R_\lambda f$. Note that $R_\lambda f = R_0 h$ and $\|h\|_{p_0} \leq 2\|f\|_{p_0}$. Hence for $\eta < \eta_0$, by [2, Proposition 5.2]

$$\|\mathcal{B}R_\lambda f\|_{p_0} = \|\mathcal{B}R_0 h\|_{p_0} \leq \frac{1}{4}\|h\|_{p_0} \leq \frac{1}{2}\|f\|_{p_0}.$$

Moreover by [2, Proposition 2.2],

$$\|R_\lambda f\|_\infty \leq c_2\|f\|_{p_0}.$$

It follows from [2, Proposition 6.1] that

$$S_\lambda f = R_\lambda\Big(\sum_{i=0}^\infty (\mathcal{B}R_\lambda)^i\Big)f$$

for $f \in L^{p_0}$ and therefore

$$\|S_\lambda f\|_\infty = \Big\|R_\lambda\Big(\sum_{i=0}^\infty (\mathcal{B}R_\lambda)^i\Big)f\Big\|_\infty \leq c_2\Big\|\Big(\sum_{i=0}^\infty (\mathcal{B}R_\lambda)^i\Big)f\Big\|_{p_0} \leq 2c_2\|f\|_{p_0}.$$

If we apply this to $f = 1_C$, where $C \subset B$, then

$$\mathbb{E}^x\left[\int_0^\infty e^{-\lambda t} 1_C(X_t)\, dt\right] \leq 2c_2|C|^{1/p_0}. \tag{2.1}$$

Let $M = \sup_{x\in B} \mathbb{E}^x\left[\int_0^\tau 1_C(X_s)\, ds\right]$. Clearly $M \leq \sup_{x\in B}\mathbb{E}^x[\tau]$, which is finite by Proposition 2.1. By taking t_1 sufficiently large,

$$\mathbb{P}^x(\tau \geq t_1) \leq \frac{\sup_{x\in B}\mathbb{E}^x[\tau]}{t_1} \leq \tfrac{1}{2}.$$

We then have

$$\begin{aligned}
\mathbb{E}^x\left[\int_0^\tau 1_C(X_s)\, ds\right] &\leq \mathbb{E}^x\Big[\int_0^{t_1} 1_C(X_s)\, ds\Big] + \mathbb{E}^x\Big[\int_{t_1}^\tau 1_C(X_s)\, ds; \tau \geq t_1\Big]\\
&\leq e^{\lambda t_1} S_\lambda 1_C(x) + \mathbb{E}^x\left[\mathbb{E}^{X_{t_1}}\Big[\int_0^\tau 1_C(X_s)\, ds\Big]; \tau \geq t_1\right]\\
&\leq c_3|C|^{1/p_0} + M\mathbb{P}^x(\tau \geq t_1).
\end{aligned}$$

Taking the supremum over x, we have

$$M \leq c_3|C|^{1/p_0} + \tfrac{1}{2}M,$$

and our result follows. □

We now prove a support theorem for X. First we prove some lemmas.

Lemma 2.3 *Let $x_0 \in \mathbb{R}^d$, $1 \leq k \leq d$, $v_k = A(x_0)e_k$, $\gamma \in (0,1)$, $t_0 > 0$, and $r \in [-1,1]$. There exists c_1 depending only on γ, t_0, r, and the upper bounds and modulus of continuity of $A(\cdot)$ in $B(x_0, 2)$ such that*

$$\begin{aligned}
\mathbb{P}^{x_0}\big(&\text{there exists a stopping time } T \leq t_0 \text{ such that}\\
&\sup_{s<T}|X_s - x_0| < \gamma \text{ and } \sup_{T\leq s\leq t_0}|X_s - (x_0 + rv_k)| < \gamma\big) \geq c_1.
\end{aligned} \tag{2.2}$$

Proof. Let $\|A\|_\infty := 1 \vee \left(\sum_{i,j=1}^d \sup_{x\in B(x_0,2)} |A_{ij}(x)|\right)$. We do the case where $r \geq 0$, the other case being similar. We first suppose $r \geq \gamma/3$. Let $\beta \in (0,r)$ be chosen later, let

$$\widetilde{Z}_t^i = \sum_{s\leq t} \Delta Z_s^i 1_{(|\Delta Z_s^i|>\beta)}, \qquad \overline{Z}_t^i = Z_t^i - \widetilde{Z}_t^i,$$

and let $\overline{X}$ be the solution to

$$d\overline{X}_s = A(\overline{X}_{s-})\, d\overline{Z}_s, \qquad \overline{X}_0 = x_0.$$

Choose $\delta < \gamma/(6\|A\|_\infty)$ such that

$$\sup_{i,j} \sup_{|x-x_0|<\delta} |A_{ij}(x) - A_{ij}(x_0)| < \gamma/(12d). \tag{2.3}$$

Let

$$\begin{aligned}
C &= \left\{ \sup_{s\leq t_0} |\overline{X}_s - \overline{X}_0| \leq \delta \right\}, \\
D &= \{\widetilde{Z}_s^i = 0 \text{ for all } s \leq t_0 \text{ and } i \neq k,\ \widetilde{Z}_k \text{ has a single jump before time } t_0 \\
&\qquad \text{and its size is in } [r, r+\delta]\}, \\
E &= \{\widetilde{Z}_s^i = 0 \text{ for all } s \leq t_0 \text{ and } i = 1, \ldots, d\}.
\end{aligned}$$

As in the proof of Proposition 2.1,

$$\mathbb{E}\,[\overline{X}^i, \overline{X}^i]_t \leq c_2 \sum_{j=1}^d \mathbb{E}\,[\overline{Z}^j, \overline{Z}^j]_t \leq c_3 t \beta^{2-\alpha},$$

and by Chebyshev's inequality and Doob's inequality,

$$\begin{aligned}
\mathbb{P}\left(\sup_{s\leq t_0} |\overline{X}_s^i - \overline{X}_0^i| > \delta/\sqrt{d} \right) &\leq \frac{\mathbb{E}\left[\sup_{s\leq t_0} \left(\overline{X}_s^i - \overline{X}_0^i \right)^2 \right]}{\delta^2/d} \\
&\leq \frac{4\mathbb{E}\left[\left(\overline{X}_{t_0}^i - \overline{X}_0^i \right)^2 \right]}{\delta^2/d} \leq \frac{c_4 t_0 \beta^{2-\alpha}}{\delta^2}.
\end{aligned}$$

We choose $\beta < r$ so that

$$c_4 t_0 \beta^{2-\alpha} \leq \delta^2/(2d), \tag{2.4}$$

and then $\mathbb{P}^{x_0}(C) \geq 1/2$.

In order for $\widetilde{Z}^k$ to have a single jump before time t_0, and for that jump's size to be in the interval $[r, r+\delta]$, then by time t_0, (a) $\widetilde{Z}^k$ must have no negative jumps; (b) $\widetilde{Z}^k$ must have no jumps whose size lies in $[\beta, r)$; (c) $\widetilde{Z}^k$ must have no jumps whose size lies in $(r+\delta, \infty)$; and (d) $\widetilde{Z}^k$ must have precisely one jump whose size lies in the interval $[r, r+\delta]$. The events described in (a)–(d) are independent and are the probabilities that Poisson random variables of parameters $c_5 t_0 \beta^{-\alpha}$, $c_5 t_0 (\beta^{-\alpha} - r^{-\alpha})$, $c_5 t_0 (r+\delta)^{-\alpha}$, and $c_5 t_0 (r^{-\alpha} - (r+\delta)^{-\alpha})$, respectively, take the values 0, 0, 0, and 1, respectively. For $j \neq k$, the probability that $\widetilde{Z}^j$ does not have a jump before time t_0 is the probability that a Poisson random variable with parameter $2c_5 t_0 \beta^{-\alpha}$ is equal to 0. Since the $\widetilde{Z}^j$, $j = 1, \cdots, d$, are independent, we thus see that the probability of D is bounded below by a constant depending on r, δ, t_0 and β. Because the $\overline{Z}_t$'s are independent of the $\widetilde{Z}^j$'s, then C and D are independent. Therefore

$$\mathbb{P}^{x_0}(C \cap D) \geq c_6/2. \tag{2.5}$$

A similar (but slightly easier) argument shows that

$$\mathbb{P}^{x_0}(C \cap E) \geq c_7. \tag{2.6}$$

If T is the time when $\widetilde{Z}^k$ jumps, then $Z_{s-} = \overline{Z}_{s-}$ for $s \leq T$, and hence $X_{s-} = \overline{X}_{s-}$ for $s \leq T$. So up to time T, X_s does not move more than δ away from its starting point. We have

$$\Delta X_T = A(X_{T-})\Delta Z_T,$$

so using (2.3) we have that on $C \cap D$,

$$\begin{aligned} &|X_T - (x_0 + rv_k)| \\ &\qquad \leq |X_{T-} - x_0| + |\Delta X_T - rv_k| \\ &\qquad = |X_{T-} - x_0| + |A(X_{T-})\Delta Z_T - rv_k| \\ &\qquad \leq |X_{T-} - x_0| + r|(A(X_{T-}) - A(x_0))e_k| + |A(X_{T-})(\Delta Z_T - re_k)| \\ &\qquad \leq \delta + rd\gamma/(12d) + \delta\|A\|_\infty < \gamma/2. \end{aligned}$$

We now apply the strong Markov property at time T. By (2.6), $\mathbb{P}^{X_T}(C \cap E) \geq c_7$ and so

$$\mathbb{P}\left(\sup_{T \leq s \leq T + t_0} |X_s - X_T| < \delta \right) \geq c_8.$$

Using the strong Markov property, we have our result with $c_1 = c_7 c_8/2$.

If $r < \gamma/3$, the argument is easier. In this case we can take T identically 0, and use (2.6). The details are left to the reader. □

Lemma 2.4 *Suppose u, v are two vectors in $\mathbb{R}^d$, $\eta \in (0,1)$, and p is the projection of v onto u. If $|p| \geq \eta|v|$, then*

$$|v - p| \leq \sqrt{1 - \eta^2}\,|v|.$$

Proof. Note that the vector $v - p$ is orthogonal to the vector p. So by the Pythagorean theorem, $|v - p|^2 = |v|^2 - |p|^2 \leq (1 - \eta^2)|v|^2$. □

Lemma 2.5 *Suppose the entries of A and A^{-1} are bounded by Λ. Let v be a vector in $\mathbb{R}^d$, $u_k = Ae_k$, and p_k the projection of v onto u_k for $k = 1, \ldots, d$. Then there exists $\rho \in (0,1)$ depending only on Λ such that for some k,*

$$|v - p_k| \leq \rho|v|.$$

Proof. Since the entries of A^{-1} are bounded, then $|(A^T)^{-1}w| \leq c_1|w|$. Setting $x = (A^T)^{-1}w$, we see $|A^T x| \geq c_2|x|$ for any vector x.

Let b_k be the projection of $A^T v$ onto e_k. If $|b_k| < (1/d)|A^T v|$ for all k, then

$$|A^T v| = \Big|\sum_{k=1}^d b_k\Big| \le \sum_{k=1}^d |b_k| < |A^T v|,$$

a contradiction. So for some k, $|b_k| \ge (1/d)|A^T v| \ge c_3|v|$, where $c_3 = c_2/d$. We then write

$$c_3|v| \le |b_k| = |v^T A e_k| \le \frac{c_4}{|Ae_k|}\,|v^T A e_k| = c_4\frac{|v^T u_k|}{|u_k|} = c_4|p_k|.$$

We now apply Lemma 2.4 with $\eta = c_3/c_4$ and set $\rho = \sqrt{1-(c_3/c_4)^2}$. □

Lemma 2.6 *Suppose the entries of $A(x)$ and $A(x)^{-1}$ on $B(x_0,3)$ are bounded by Λ. Let $t_1 > 0$, $\varepsilon \in (0,1)$, $r \in (0,\varepsilon/4)$ and $\gamma > 0$. Let $\psi : [0,t_1] \to \mathbb{R}^d$ be a line segment of length r starting at x_0. Then there exists $c_1 > 0$ that depends only on Λ, the modulus of continuity of $A(x)$ on $B(x_0,3)$, t_1, ε and γ such that*

$$\mathbb{P}^{x_0}\left(\sup_{s\le t_1}|X_s - \psi(s)| < \varepsilon \ \text{ and } \ |X_{t_1} - \psi(t_1)| < \gamma\right) \ge c_1.$$

Proof. Use the bounds on A in $B(x_0,2)$ and Lemma 2.5 to define $\rho \in (0,1)$ so that the conclusion of Lemma 2.5 holds for all matrices $A = A(x)$ with $x \in B(x_0,2)$. Take $\gamma \in (0, r\wedge\rho)$ smaller if necessary so that $\widetilde\rho := \gamma + \rho < 1$. Choose $n \ge 2$ large so that $(\widetilde\rho)^n < \gamma$.

Let $v_0 = \psi(t_1) - \psi(t_0) = \psi(t_1) - x_0$, which has length r. By Lemma 2.5, there exists $k_0 \in \{1,\cdots,d\}$ such that if p_0 is the projection of v_0 onto $A(x_0)e_{k_0}$, then $|v_0 - p_0| \le \rho|v_0|$. Note $|p_0| \le |v_0| = r$.

Let D_1 denote the event that there is a stopping time $T_0 \le t_1/n$ such that $|X_s - x_0| < \gamma^{n+1}$ for $s < T_0$ and $|X_s - (x_0+p_0)| < \gamma^{n+1}$ for $s \in [T_0,\, t_1/n]$. By Lemma 2.3 there exists $c_2 > 0$ such that $\mathbb{P}^{x_0}(D_1) \ge c_2$. Note that on D_1, if $T_0 \le s \le t_1/n$,

$$|\psi(t_1) - X_s| \le |\psi(t_1) - (x_0+p_0)| + |(x_0+p_0) - X_{t_1/n}| \le \rho r + \gamma^{n+1} \le \widetilde\rho\, r. \tag{2.7}$$

Taking $s = t_1/n$, we have

$$|\psi(t_1) - X_{t_1}| \le \widetilde\rho r.$$

Since $\widetilde\rho < 1$ and $|\psi(t_1) - x_0| = r$, then (2.7) shows that on D_1,

$$X_s \in B(x_0,2r) \subset B(x_0,\varepsilon/2) \qquad \text{if } T_0 \le s \le t_1/n.$$

If $0 \le s < T_0$, then $|X_s - x_0| < \gamma^{n+1} < r$, and so $\{X_s, s\in[0,t_1/n]\} \subset B(x_0,2r) \subset B(x_0,\varepsilon/2)$.

Now let $v_1 = \psi(t_1) - X_{t_1/n}$. When $X_{t_1/n} \in B(x_0,\varepsilon/2)$, by Lemma 2.5, there exists k_1 such that if p_1 is the projection of v_1 onto $A(X_{t_1/n})e_{k_1}$, then $|v_1 - p_1| \le \rho|v_1|$. Let D_2 be the event that there exists a stopping time $T_1 \in [t_1/n, 2t_1/n]$ such that $|X_s - X_{t_1/n}| < \gamma^{n+1}$ for $t_1/n \le s < T_1$ and $|X_s - (X_{t_1/n} + p_1)| < \gamma^{n+1}$ for $T_1 \le s \le 2t_1/n$. Using the Markov property at time t_1/n and applying Lemma 2.3 again, there exists (the same) $c_2 > 0$ such that

$$\mathbb{P}^{x_0}(D_2 \mid \mathcal{F}_{t_1/n}) \ge c_2$$

on the event $\{X_{t_1/n} \in B(x_0,\varepsilon/2)\}$, where $\mathcal{F}_t$ is the minimal augmented filtration for X. So

$$\mathbb{P}^{x_0}(D_1 \cap D_2) \ge c_2\mathbb{P}^{x_0}(D_1) \ge c_2^2.$$

On the event $D_1 \cap D_2$, if $T_1 \le s \le 2t_1/n$,

$$\begin{aligned} |\psi(t_1) - X_s| &\le |\psi(t_1) - (X_{t_1/n} + p_1)| + |(X_{t_1/n} + p_1) - X_s| \\ &\le \rho|v_1| + \gamma^{n+1} \le \rho\widetilde{\rho}\, r + \gamma^{n+1} \le \widetilde{\rho}^2\, r. \end{aligned}$$

In particular

$$|\psi(t_1) - X_{2t_1/n}| \le \widetilde{\rho}^2 r \qquad \text{on } D_1 \cap D_2.$$

If $T_1 \le s \le 2t_1/n$, then $|\psi(t_1) - X_s| < r$ and $|\psi(t_1) - x_0| = r$, and so $X_s \in B(x_0, 2r) \subset B(x_0, \varepsilon/2)$. In particular,

$$|X_{2t_1/n} - x_0| < \varepsilon/2 \qquad \text{on } D_1 \cap D_2.$$

If $t_1/n \le s < T_1$, then $|X_s - X_{t_1/n}| < r$ and $|X_{t_1/n} - x_0| < 2r$. So on $D_1 \cap D_2$, $X_s \in B(x_0, 3r) \subset B(x_0, 3\varepsilon/4)$.

Let $v_2 = \psi(t_1) - X_{2t_1/n}$, and proceed as above to get events $D_3, \cdots, D_k$. At the k^{th} stage, we have

$$\mathbb{P}^{x_0}(D_k \mid \mathcal{F}_{(k-1)t_1/n}) \ge c_2$$

and so $\mathbb{P}^{x_0}(\cap_{j=1}^k D_j) \ge c_2^k$. On the event $\cap_{j=1}^k D_j$, if $kt_1/n \le T_k \le s \le (k+1)t_1/n$, then

$$|\psi(t_1) - X_s| \le \widetilde{\rho}^{k+1} r < r;$$

since $|\psi(t_1) - x_0| = r$, then $X_s \in B(x_0, 2r) \subset B(x_0, \varepsilon/2)$. At the k^{th} stage, on the event $\cap_{j=1}^k D_j$,

$$|X_{kt_1/n} - x_0| < \varepsilon/2$$

and if $kt_1/n \le s < T_k$, then

$$|X_s - x_0| \le |X_s - X_{kt_1/n}| + |X_{kt_1/n} - \psi(t_1)| + |\psi(t_1) - x_0| < \gamma^{n+1} + 2r + r < 3r,$$

and so $X_s \in B(x_0, 3r) \subset B(x_0, 3\varepsilon/4)$.

We continue this procedure n times to get events $D_1, \cdots, D_n$ so that on $\cap_{k=1}^n D_k$, we have $X_s \in B(x_0, 3\varepsilon/4)$ for $s \le t_1$, $|X_{t_1} - \psi(t_1)| < \gamma$, and $\mathbb{P}^{x_0}(\cap_{k=1}^n D_k) \ge c_2^n$. Consequently, on $\cap_{k=1}^n D_k$,

$$|X_s - \psi(s)| \le |X_s - x_0| + |x_0 - \psi(s)| < 3\varepsilon/4 + r < \varepsilon \qquad \text{for } s \in [0, t_1].$$

This completes the proof of the lemma. □

Theorem 2.7 *Suppose the entries of $A(x)$ and $A(x)^{-1}$ on $B(x_0, 3)$ are bounded by Λ. Let $\varphi : [0, t_0] \to \mathbb{R}^d$ be continuous with $\varphi(0) = x_0$ and the image of φ contained in $B(0, 1)$. Let $\varepsilon > 0$. There exists $c_1 > 0$ depending on Λ, the modulus of continuity of $A(x)$ on $B(x_0, 3)$, φ, ε, and t_0 such that*

$$\mathbb{P}^{x_0}\left(\sup_{s \le t_0} |X_s - \varphi(s)| < \varepsilon\right) > c_1.$$

Proof. We may approximate φ to within $\varepsilon/2$ by a polygonal path, so by changing ε to $\varepsilon/2$, we may without loss of generality assume φ is polygonal. Let us now choose n large and subdivide $[0, t_0]$ into n equal subintervals so that over each subinterval $[kt_0/n, (k+1)t_0/n]$ the image of φ is a line segment of length less than $\varepsilon/4$. We then use Lemma 2.6 and the strong Markov property n times to show that, with probability at least $c_1 > 0$, on each time interval $[kt_0/n, (k+1)t_0/n]$, X_t follows within $\varepsilon/2$ the line segment from $X_{kt_0/n}$ to $\varphi((k+1)t_0/n)$ and is at most $\varepsilon/(4\sqrt{d})$ away from $\varphi((k+1)t_0/n)$. □

Corollary 2.8 *Let $\varepsilon, \delta \in (0, 1/4)$. Suppose Q represents either the unit ball or the unit cube, centered at $x_0 \in \mathbb{R}^d$. Suppose the entries of A and A^{-1} on Q are bounded by Λ. Let Q' be the ball (resp., cube) with radius (resp., side length) $1 - \varepsilon$ with the same center. Let R be a ball (resp., cube) of radius (resp., side length) δ contained in Q'. Then there exists $c_1 > 0$ depending on Λ, the modulus of continuity of $A(x)$ on Q, ε and δ such that*

$$\mathbb{P}^x(T_R < \tau_Q) \geq c_1, \qquad x \in Q'.$$

Proof. Note that the above probability is determined by the values of the matrix $A(x)$ only on Q so we can redefine $A(x)$ outside of Q if necessary to make the entries of A and A^{-1} on $\mathbb{R}^d$ bounded by Λ, and the modulus of continuity of $A(x)$ on $\mathbb{R}^d$ be the same as that on Q. To prove the corollary, we need only observe that the estimates in Theorem 2.7 can be made to hold uniformly over every line segment from x to y, with $x \in Q'$ and y being the center of R. □

A scaling argument shows that for $\lambda > 0$, $\{\widehat{X}_t := \lambda X_{t/\lambda^\alpha},\, t \geq 0\}$ is a process of the same type as X. More precisely, one can show that there exist d independent one-dimensional symmetric stable processes $\widehat{Z}^j$ of index α such that $\widehat{X}$ satisfies

$$d\widehat{X}_t^i = \sum_{j=1}^d \widehat{A}_{ij}(\widehat{X}_t)\, dZ_t^j, \qquad \widehat{X}_0^i = \lambda x_0^i,$$

where $\widehat{A}_{ij}(x) = A_{ij}(x/\lambda)$. Note in particular that when $\lambda \geq 1$, the oscillation of $\widehat{A}$ will be no more than the oscillation of A. A consequence is that the analogues of Propositions 2.1 and 2.2 and Theorem 2.7 hold in balls $B(x_1, r)$ with the same constants provided $r < 1$ (so that $\lambda = 1/r > 1$).

We now have what is needed to prove our main theorem.

Theorem 2.9 *Let $r \in (0, 1]$ and $\gamma > 1$. Suppose h is harmonic in $B(x_0, \gamma r)$ with respect to X and h is bounded in $\mathbb{R}^d$. There exists positive constants c_1 and β that depend on γ, the upper bound of $A(x)$ and $A(x)^{-1}$ on $B(x_0, \gamma r)$, and the modulus of continuity of $A(x)$ on $B(x_0, \gamma r)$ but otherwise is independent of h and r such that*

$$|h(x) - h(y)| \leq c_1 \Big(\frac{|x-y|}{r} \Big)^\beta \sup_{\mathbb{R}^d} |h(z)|$$

Proof. If one examines the proof of Krylov-Safonov carefully (see, e.g., the presentation in [1], Theorem V.7.4), one sees that one needs the support theorem and Corollary 2.8, a result such as Proposition 2.2 and estimates such as Proposition 2.1 and that with these ingredients, one can conclude that if Q is a cube of side length $r \leq 1$, $A \subset Q \subset B(x_0, r)$, and Q' is a cube with the same center as Q but side length half as long, then

$$\mathbb{P}^x(T_A < \tau_Q) \geq \varphi(|A|/|Q|) \qquad \text{for } x \in Q', \tag{2.8}$$

where φ is a strictly increasing function with $\varphi(0) = 0$.

Now let $B = B(y, s)$ be a ball contained in $B(x_0, r)$ and suppose $A \subset B$ with $|A|/|B| \geq 1/3$. Let $B' = B(y, (1-\varepsilon)s)$, where ε is chosen so that $|B \setminus B'|/|B| = 1/6$. Then $|A \cap B'|/|B| \geq 1/6$. Cover B' with N equally sized cubes whose interiors are disjoint and each contained in B. We may choose N independent of s. For at least one cube, say, Q, we must have $|A \cap B' \cap Q|/|Q| \geq 1/6$. Let Q' be the cube with the same center as Q but side length half as long. By the support theorem, if

$x \in B(y, s/2)$, there is probability at least c_2 such that $\mathbb{P}^x(T_{Q'} < \tau_B) \geq c_2$. Applying (2.8) and the strong Markov property, we have

$$\mathbb{P}^x(T_A < \tau_B) \geq c_3 > 0 \qquad \text{for } x \in B(y, s/2). \tag{2.9}$$

Applying (2.9) and Proposition 2.1, the result now follows exactly as the proof in Theorem 4.1 of [3]. (We remark that line 15 on page 386 of [3] should read instead

$$(b_{k-1} - a_{k-1})\mathbb{P}^y(\tau_k < T_A) \leq \frac{1}{\gamma}(b_k - a_k)(1 - \mathbb{P}^y(T_A < \tau_k)).$$

With suitable modifications to the definition of γ and ρ, the proof of Theorem 4.1 in [3] is valid.)
□

3 A counterexample to the Harnack inequality

We now show that one cannot expect a Harnack inequality to hold, even when $A(x) \equiv I$, the identity matrix. We will describe ε in a moment. Write points in $\mathbb{R}^3$ as $w = (x, y, z)$ and let $w_0 = (0, \frac{1}{2}, 0)$. Write B for $B(0, 1)$, τ for τ_B, and let $F_\varepsilon = (-\varepsilon, \varepsilon)^2 \subset \mathbb{R}^2$, $C_\varepsilon = (\mathbb{R} \times F_\varepsilon) \cap B$, and $E_\varepsilon = (2, 4) \times F_\varepsilon$. Let X_t, Y_t and Z_t be independent one-dimensional symmetric α-stable processes and set $W_t = (X_t, Y_t, Z_t)$. Define $h_\varepsilon(w) = \mathbb{P}^w(W_\tau \in E_\varepsilon)$. We will show that $h_\varepsilon(0)/h_\varepsilon(w_0) \to \infty$ as $\varepsilon \to 0$; this implies that a Harnack inequality is not possible.

The Lévy measure $n(w, d\widetilde{w})$ of W is

$$n(w, d\widetilde{w}) = c \sum_{k=1}^{3} |w_k - \widetilde{w}_k|^{-1-\alpha}\, d\widetilde{w}_k \prod_{j \neq k} \delta_{w_j}(d\widetilde{w}_j)$$

where δ_a denotes the Dirac measure at the point a. Since all jumps of W are in directions parallel to the coordinate axes, the only way W_τ can be in E_ε is if $W_{\tau-}$ is in C_ε. This is the key observation.

We first get an upper bound on h_ε. It is well known that if $p_t(u, v)$ is the transition density for a one-dimensional symmetric stable process of index α, then p_t is everywhere strictly positive, is jointly continuous, $p_t(u, v) = t^{-1/\alpha} p_1(u/t^{1/\alpha}, v/t^{1/\alpha})$, and $p_1(u, v) \sim c_1 |u - v|^{-\alpha-1}$ for $|u - v|$ large. An integration gives

$$\begin{aligned} &\mathbb{E}^{(y,z)} \left[\int_0^\infty 1_{(-1,1)^2}(Y_s, Z_s)\, ds \right] \\ &\qquad \leq 1 + \int_1^\infty \Big(\int_{-1}^1 p_t(y, u)\, du \Big) \Big(\int_{-1}^1 p_t(z, v)\, dv \Big)\, ds < \infty. \end{aligned}$$

By scaling,

$$\mathbb{E}^{(y,z)} \left[\int_0^\infty 1_{F_\varepsilon}(Y_s, Z_s)\, ds \right] < c_2 \varepsilon^\alpha.$$

By the Lévy system formula (see [3] or [5]),

$$\begin{aligned}\mathbb{E}^{w}\left[\sum_{s\le t\wedge\tau} 1_{(W_{s-}\in C_\varepsilon, W_s\in E_\varepsilon)}\right] &= \mathbb{E}^{w}\left[\int_0^{t\wedge\tau} 1_{C_\varepsilon}(W_s) n(W_s, E_\varepsilon)\, ds\right]\\ &\le c_3 \mathbb{E}^{w}\left[\int_0^{\infty} 1_{C_\varepsilon}(W_s)\, ds\right]\\ &\le c_3 \mathbb{E}^{(y,z)}\left[\int_0^{\infty} 1_{F_\varepsilon}(Y_s, Z_s)\, ds\right]\\ &\le c_2 c_3 \varepsilon^\alpha. \end{aligned} \tag{3.1}$$

Letting $t\to\infty$, we obtain

$$h_\varepsilon(w) = \mathbb{P}^w(W_\tau\in E_\varepsilon) \le c_4\varepsilon^\alpha. \tag{3.2}$$

Next we get a lower bound on $h_\varepsilon(0)$. Let $C'_\varepsilon = C_\varepsilon \cap \{|x|<1/2\}$. By the Lévy system formula we have

$$\begin{aligned} h_\varepsilon(0) &\ge \mathbb{E}^{0}\left[\sum_{s\le t\wedge\tau} 1_{(W_{s-}\in C'_\varepsilon, W_s\in E_\varepsilon)}\right]\\ &= \mathbb{E}^{0}\left[\int_0^{t\wedge\tau} 1_{C'_\varepsilon}(W_s) n(W_s, E_\varepsilon)\, ds\right]\\ &\ge c_5 \mathbb{E}^{0}\left[\int_0^{t\wedge\tau} 1_{C'_\varepsilon}(W_s)\, ds\right]. \end{aligned}$$

Letting $t\to\infty$,

$$h_\varepsilon(0) \ge c_5 \mathbb{E}^{0}\left[\int_0^{\tau} 1_{C'_\varepsilon}(W_s)\, ds\right].$$

By the scaling property of α-stable processes, if $\overline{V}$ is a one-dimensional symmetric α-stable process starting from 0 killed on exiting $[-1/4, 1/4]$, then $\varepsilon^{-1}V_t$ has the same distribution as $\overline{U}_{t/\varepsilon^\alpha}$, where $\overline{U}$ is a one-dimensional symmetric α-stable process starting from 0 killed on exiting $[-1/(4\varepsilon), 1/(4\varepsilon)]$. Hence there is a positive constant $c_6>0$ such that for every $\varepsilon\in(0,1)$ and $t\in(0,\varepsilon^\alpha)$,

$$\mathbb{P}^0(\overline{V}_t\in[-\varepsilon,\varepsilon]) = \mathbb{P}^x(\overline{U}_{t/\varepsilon^\alpha}\in[-1,1]) \ge c_6.$$

Consequently,

$$\mathbb{E}^{0}\left[\int_0^{\infty} 1_{C'_\varepsilon}(W_s)\, ds\right] \ge \mathbb{E}^{0}\left[\int_0^{\varepsilon^\alpha} 1_{C'_\varepsilon}(\overline{W}_s)\, ds\right] \ge c_7\varepsilon^\alpha,$$

where $\overline{W}$ is the process W killed when any of X, Y, or Z exceeds $1/4$ in absolute value. Therefore

$$h_\varepsilon(0)\ge c_8\varepsilon^\alpha. \tag{3.3}$$

Let $G=(-1,1)^2\subset\mathbb{R}^2$, write $\widehat{w}$ for (y,z), and $\widehat{W}_t=(Y_t,Z_t)$. By the estimates on the transition densities, we see that

$$u(\widehat{w}) := \mathbb{E}^{\widehat{w}}\left[\int_0^\infty 1_G(\widehat{W}_s) ds\right]$$

is bounded and

$$u(\widehat{w}) \le \int_0^\infty \mathbb{P}^y(|Y_s|<1)\mathbb{P}^z(|Z_s|<1)\, ds \to 0 \tag{3.4}$$

as $|\widehat{w}| \to \infty$. Similarly, for $\widehat{w} \in G$,

$$u(\widehat{w}) \geq \int_1^2 \mathbb{P}^y(|Y_s| < 1)\mathbb{P}^z(|Z_s| < 1)\, ds \geq c_9.$$

Now $u(\widehat{W}_{t \wedge T_B})$ is a bounded supermartingale, so by optional stopping

$$u(\widehat{w}) \geq \mathbb{E}^{\widehat{w}}[u(\widehat{W}_{T_G}); T_G < \infty] \geq c_9 \mathbb{P}^w(T_G < \infty),$$

and (3.4) then implies that $\mathbb{P}^{\widehat{w}}(T_G < \infty) \to 0$ as $\widehat{w} \to \infty$. Scaling then shows that

$$\mathbb{P}^{(1/2,0)}(T_{F_\varepsilon} < \infty) \to 0 \qquad \text{as } \varepsilon \to 0,$$

and hence

$$\mathbb{P}^{w_0}(T_{C_\varepsilon} < \infty) \to 0 \qquad \text{as } \varepsilon \to 0. \tag{3.5}$$

Therefore by (3.1)-(3.2),

$$\begin{aligned} h_\varepsilon(w_0) &= \mathbb{E}^{w_0}[h_\varepsilon(W_{T_{C_\varepsilon}}); T_{C_\varepsilon} < \tau] \\ &\leq c_{10} \varepsilon^\alpha \mathbb{P}^{w_0}(T_{C_\varepsilon} < \tau) \\ &\leq c_{11}\, h_\varepsilon(0)\, \mathbb{P}^{w_0}(T_{C_\varepsilon} < \infty). \end{aligned}$$

This and (3.5) shows that $h_\varepsilon(0)/h_\varepsilon(w_0)$ can be made as large as we like by taking ε small enough and so a Harnack inequality for W is not possible.

Remark. When $\alpha < 1$, we can construct a two-dimensional example along the same lines.

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