

RELEVANT SAMPLING OF BAND-LIMITED FUNCTIONS

RICHARD F. BASS AND KARLHEINZ GRÖCHENIG

ABSTRACT. We study the random sampling of band-limited functions of several variables. If a band-limited function with bandwidth has its essential support on a cube of volume R^d , then $\mathcal{O}(R^d \log R^d)$ random samples suffice to approximate the function up to a given error with high probability.

1. Introduction

The nonuniform sampling of band-limited functions of several variables remains a challenging problem. Whereas in dimension 1 the density of a set essentially characterizes sets of stable sampling [14], in higher dimensions the density is no longer a decisive property of sets of stable sampling. Only a few strong and explicit sufficient conditions are known, for example, [3], [10], [12].

This difficulty is one of the reasons for taking a probabilistic approach to the sampling problem [2], [20]. At first glance, one would guess that every reasonably homogeneous set of points in \mathbb{R}^d satisfying Landau's necessary density condition will generate a set of stable sampling. This intuition is far from true. To the best of our knowledge, every construction in the literature of sets of random points in \mathbb{R}^d contains either arbitrarily large holes with positive probability or concentrates near the zero manifold of a band-limited function. Both properties are incompatible with a sampling inequality. See [2] for a detailed discussion.

The difficulties with the probabilistic approach lie in the unboundedness of the configuration space \mathbb{R}^d and the infinite dimensionality of the space of band-limited functions. To resolve this issue, we argued in [2] that usually one observes only finitely many samples of a band-limited function and that these

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observations are drawn from a bounded subset of \mathbb{R}^d . Moreover, since it does not make sense to sample a given function f in a region where f is small, we proposed to sample f only on its essential support. Since f is sampled only in the relevant region, this method might be called the “relevant sampling of band-limited functions.” In this paper, we continue our investigation of the random sampling of band-limited functions and settle a question that was left open in [2], namely how many random samples are required to approximate a band-limited function locally to within a given accuracy?

To fix terms, recall that the space of band-limited functions is defined to be

$$\mathcal{B} = \{f \in L^2(\mathbb{R}^d) : \text{supp } \hat{f} \subseteq [-1/2, 1/2]^d\},$$

where we have normalized the spectrum to be the unit cube and the Fourier transform is normalized as $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx$. A set $\{x_j : j \in J\} \subseteq \mathbb{R}^d$ is called a set of stable sampling or simply a set of sampling [7], if there exist constants $A, B > 0$, such that a *sampling inequality* holds:

$$(1) \quad A\|f\|_2^2 \leq \sum_j |f(x_j)|^2 \leq B\|f\|_2^2, \quad \forall f \in \mathcal{B}.$$

Next, we sample only on the essential support of f . Therefore, we let $C_R = [-R/2, R/2]^d$ and define the subset

$$\mathcal{B}(R, \delta) = \left\{ f \in \mathcal{B} : \int_{C_R} |f(x)|^2 dx \geq (1 - \delta)\|f\|_2^2 \right\}.$$

As a continuation of [2], we will prove the following sampling theorem.

THEOREM 1. *Let $\{x_j : j \in \mathbb{N}\}$ be a sequence of independent and identically distributed random variables that are uniformly distributed in C_R . Suppose that $R \geq 2$, that $\delta \in (0, 1)$ and $\nu \in (0, 1/2)$ are small enough, and that $0 < \varepsilon < 1$. There exists a constant κ so that if the number of samples r satisfies*

$$(2) \quad r \geq 2R^d \frac{1 + \nu/3}{\nu^2} \log \frac{2R^d}{\varepsilon},$$

then the sampling inequality

$$(3) \quad \begin{aligned} & \frac{r}{R^d} \left(\frac{1}{2} - \delta - \nu - 12\delta\kappa \right) \|f\|_2^2 \\ & \leq \sum_{j=1}^r |f(x_j)|^2 \leq r\|f\|_2^2 \quad \text{for all } f \in \mathcal{B}(R, \delta) \end{aligned}$$

holds with probability at least $1 - \varepsilon$. The constant κ can be taken to be $\kappa = e^{d\pi}$.

The formulation of Theorem 1 is similar to [2, Theorem 3.1]. The main point is that only $\mathcal{O}(R^d \log R^d)$ samples are required for a sampling inequality to hold with high probability. In [2], we used a metric entropy argument to

show that $\mathcal{O}(R^{2d})$ samples suffice. We expect that the order $\mathcal{O}(R^d \log R^d)$ is optimal. We point out that in addition all constants are now explicit.

Our idea is to replace the sampling of band-limited function in $\mathcal{B}(R, \delta)$ by a finite-dimensional problem, namely the sampling of the corresponding span of prolate spheroidal functions on the cube $[-R/2, R/2]^d$ and then use error estimates. For the probability estimates we use a new tool, namely the powerful matrix Bernstein inequality of Ahlswede and Winter [1] in the optimized version of Tropp [22].

The remainder of the paper contains the analysis of a related finite-dimensional problem for prolate spheroidal functions in Section 2 and transition to the infinite-dimensional problem in $\mathcal{B}(R, \delta)$ with the necessary error estimates in Section 3. The Appendix contains an elementary estimate for the constant κ .

2. Finite-dimensional subspaces of \mathcal{B}

We first study a sampling problem in a finite-dimensional subspace related to the set $\mathcal{B}(R, \delta)$.

Prolate spheroidal functions. Let P_R and Q be the projection operators defined by

$$(4) \quad P_R f = \chi_{C_R} f \quad \text{and} \quad Q f = \mathcal{F}^{-1}(\chi_{[-1/2, 1/2]^d} \hat{f}),$$

where \mathcal{F}^{-1} is the inverse Fourier transform. The composition of these orthogonal projections

$$(5) \quad A_R = Q P_R Q$$

is the operator of time and frequency limiting. This operator arises frequently in the context of band-limited functions and uncertainty principles. The localization operator A_R is a compact positive operator of trace class, and by results of Landau, Slepian, Pollak and Widom [8], [9], [17], [18], [19], [24] the eigenvalue distribution spectrum is precisely known. We summarize the properties of the spectrum that we will need.

Let $A_R^{(1)}$ denote the operator of time-frequency limiting in dimension $d = 1$. This operator can be defined explicitly on $L^2(\mathbb{R})$ by the formula

$$(A_R^{(1)} f)^\wedge(\xi) = \int_{-1/2}^{1/2} \frac{\sin \pi R(\xi - \eta)}{\pi(\xi - \eta)} \hat{f}(\eta) d\eta \quad \text{for } |\xi| \leq 1/2.$$

The eigenfunctions of $A_R^{(1)}$ are the prolate spheroidal functions, and let the corresponding eigenvalues $\mu_k = \mu_k(R)$ be arranged in decreasing order. According to [6], they satisfy

$$0 < \mu_k(R) < 1 \quad \forall k \in \mathbb{N},$$

$$\mu_{[R]+1}(R) \leq 1/2 \leq \mu_{[R]-1}(R).$$

As a consequence any function with spectrum $[-1/2, 1/2]$ and “essential” support on $[-R/2, R/2]$ is close to the span of the first R prolate spheroidal functions. In particular, we may think of $\mathcal{B}(R, \delta)$ as, roughly, almost a subset of a finite-dimensional space of dimension R .

The time-frequency limiting operator A_R on $L^2(\mathbb{R}^d)$ is the d -fold tensor product of $A_R^{(1)}$, $A_R = A_R^{(1)} \otimes \cdots \otimes A_R^{(1)}$. Consequently, $\sigma(A_R)$, the spectrum of A_R , is

$$\sigma(A_R) = \left\{ \lambda \in (0, 1) : \lambda = \prod_{j=1}^d \mu_{k_j}, \mu_{k_j} \in \sigma(A_R^{(1)}) \right\}.$$

Since $0 < \mu_k < 1$, A_R possesses at most R^d eigenvalues greater than or equal to $1/2$. Again we arrange the eigenvalues of A_R by magnitude $1 > \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \geq \lambda_{n+1} \geq \cdots > 0$. Let ϕ_j be the eigenfunction corresponding to λ_j .

We fix R “large” and $\delta \in (0, 1)$. Let

$$\mathcal{P}_N = \text{span}\{\phi_j : j = 1, \dots, N\}$$

be the span of the first N eigenfunctions of the time-frequency limiting operator A_R (one might call functions in \mathcal{P}_N “multivariate prolate polynomials”). For properly chosen N , \mathcal{P}_N consists of functions in $\mathcal{B}(R, \delta)$. See Lemma 5.

By Plancherel’s theorem,

$$\langle Qf, g \rangle = \langle \chi_{[-1/2, 1/2]^d} \hat{f}, \hat{g} \rangle = \langle \hat{f}, \chi_{[-1/2, 1/2]^d} \hat{g} \rangle = \langle f, Qg \rangle.$$

Then for $f \in \mathcal{B}$ we have $Qf = f$, and so

$$(6) \quad \langle A_R f, f \rangle = \langle P_R Qf, Qf \rangle = \langle P_R f, f \rangle = \int_{C_R} |f(x)|^2 dx.$$

We first study random sampling in the finite-dimensional space \mathcal{P}_N . In the following $\|f\|_{2,R}$ denotes the normalized L^2 -norm of f restricted to the cube $C_R = [-R/2, R/2]^d$:

$$\|f\|_{2,R}^2 = \int_{C_R} |f(x)|^2 dx.$$

PROPOSITION 2. *Let $\{x_j : j \in \mathbb{N}\}$ be a sequence of independent and identically distributed random variables that are uniformly distributed in $[-R/2, R/2]^d$. Then*

$$(7) \quad \mathbb{P} \left(\inf_{f \in \mathcal{P}_N, \|f\|_2=1} \frac{1}{r} \sum_{j=1}^r \left(|f(x_j)|^2 - \frac{1}{R^d} \|f\|_{2,R}^2 \right) \leq -\frac{\nu}{R^d} \right) \\ \leq N \exp \left(-\frac{\nu^2 r/2}{R^d(1+\nu/3)} \right)$$

for $r \in \mathbb{N}$ and $\nu \geq 0$.

Proof. We prove the proposition in several steps. First, since \mathcal{P}_N is finite-dimensional, the sampling inequality for \mathcal{P}_N amounts to a statement about the spectrum of an underlying (random) matrix.

Let $f = \langle c, \phi \rangle = \sum_{k=1}^N c_k \phi_k \in \mathcal{P}_N$, so that $|f(x_j)|^2 = \sum_{k,l=1}^N c_k \bar{c}_l \phi_k(x_j) \times \overline{\phi_l(x_j)}$. Now define the $N \times N$ matrix T_j of rank one by letting the (k, l) entry be

$$(8) \quad (T_j)_{kl} = \phi_k(x_j) \overline{\phi_l(x_j)}.$$

Then $|f(x_j)|^2 = \langle c, T_j c \rangle$. Since each random variable x_j is uniformly distributed over C_R and ϕ_k is the k th eigenfunction of the localization operator A_R , using (6) the expectation of the kl th entry is

$$(9) \quad \begin{aligned} \mathbb{E}((T_j)_{kl}) &= \frac{1}{R^d} \int_{C_R} \phi_k(x) \overline{\phi_l(x)} dx \\ &= \frac{1}{R^d} \langle A_R \phi_k, \phi_l \rangle \\ &= \frac{1}{R^d} \lambda_k \delta_{kl}, \quad k, l = 1, \dots, N, \end{aligned}$$

where δ_{kl} is Kronecker's delta. Consequently the expectation of T_j is the diagonal matrix

$$(10) \quad \mathbb{E}(T_j) = \frac{1}{R^d} \text{diag}(\lambda_k) =: \frac{1}{R^d} \Delta.$$

We may now rewrite the expression in (7) as

$$(11) \quad \begin{aligned} &\inf_{f \in \mathcal{P}_N, \|f\|_2=1} \frac{1}{r} \sum_{j=1}^r \left(|f(x_j)|^2 - \frac{1}{R^d} \|f\|_{2,R}^2 \right) \\ &= \inf_{\|c\|_2=1} \frac{1}{r} \sum_{j=1}^r (\langle c, T_j c \rangle - \langle c, \mathbb{E}(T_j) c \rangle) \\ &= \lambda_{\min} \left(\frac{1}{r} \sum_{j=1}^r (T_j - \mathbb{E}(T_j)) \right), \end{aligned}$$

where we use $\lambda_{\min}(U)$ for the smallest eigenvalue of a self-adjoint matrix U .

Consequently, we have to estimate a probability for the matrix norm of a sum of random matrices. We do this using a matrix Bernstein inequality due to Tropp [22]. Let $\lambda_{\max}(A)$ be the largest singular value of a matrix A so that $\|A\| = \lambda_{\max}(A^* A)^{1/2}$ is the operator norm (with respect to the ℓ^2 -norm).

THEOREM 3 (Tropp). *Let X_j be a sequence of independent, random self-adjoint $N \times N$ -matrices. Suppose that*

$$\mathbb{E}X_j = 0 \quad \text{and} \quad \|X_j\| \leq B \quad \text{a.s.}$$

and let

$$\sigma^2 = \left\| \sum_{j=1}^r \mathbb{E}(X_j^2) \right\|.$$

Then for all $t \geq 0$,

$$(12) \quad \mathbb{P} \left(\lambda_{\max} \left(\sum_{j=1}^r X_j \right) \geq t \right) \leq N \exp \left(-\frac{t^2/2}{\sigma^2 + Bt/3} \right).$$

To apply the matrix Bernstein inequality, we set $X_j = T_j - \mathbb{E}(T_j)$. We need to calculate $\|X_j\|$ and $\|\sum_j \mathbb{E}(X_j^2)\|$. Clearly $\mathbb{E}(X_j) = 0$.

LEMMA 4. *Under the conditions stated above, we have*

$$\begin{aligned} \|X_j\| &\leq 1, \\ \mathbb{E}(X_j^2) &\leq R^{-d} \Delta \end{aligned}$$

and

$$\sigma^2 = \left\| \sum_{j=1}^r \mathbb{E}(X_j^2) \right\| \leq \frac{r}{R^d}.$$

Proof. (i) To estimate the matrix norm of X_j , recall that

$$(13) \quad |f(x)| \leq \|f\|_2 \quad \forall f \in \mathcal{B}.$$

Hence, we obtain

$$\begin{aligned} \|X_j\| &= \sup_{\|f\|_2=1} \left| |f(x_j)|^2 - R^{-d} \|f\|_{2,R}^2 \right| \\ &\leq \|f\|_\infty - R^{-d} \|f\|_{2,R}^2 \leq \|f\|_2 = 1. \end{aligned}$$

(ii) Next, we calculate the matrix $\mathbb{E}(X_j^2)$:

$$\begin{aligned} \mathbb{E}(X_j^2) &= \mathbb{E}(T_j^2) - R^{-d} \mathbb{E}(T_j \Delta) - R^{-d} \mathbb{E}(\Delta T_j) + R^{-2d} \Delta^2 \\ &= \mathbb{E}(T_j^2) - R^{-d} \mathbb{E}(T_j) \Delta - R^{-d} \Delta \mathbb{E}(T_j) + R^{-2d} \Delta^2 \\ &= \mathbb{E}(T_j^2) - R^{-2d} \Delta^2. \end{aligned}$$

Furthermore, the square of the rank one matrix T_j is the (rank one) matrix

$$\begin{aligned} (T_j^2)_{km} &= \sum_{l=1}^N (T_j)_{kl} (T_j)_{lm} \\ &= \sum_l \phi_k(x_j) \overline{\phi_l(x_j)} \phi_l(x_j) \overline{\phi_m(x_j)} \\ &= \left(\sum_{l=1}^N |\phi_l(x_j)|^2 \right) (T_j)_{km}. \end{aligned}$$

Writing $m(x) = \sum_{l=1}^N |\phi_l(x)|^2$, we obtain

$$(14) \quad T_j^2 = m(x_j)T_j.$$

Let s be the function whose Fourier transform is given by $\hat{s} = \chi_{[-1/2, 1/2]^d}$ and let $T_x f(t) = f(t - x)$ be the translation operator. Then it is well known that $T_x s$ is the reproducing kernel for \mathcal{B} , that is,

$$f(x) = \langle f, T_x s \rangle.$$

To see this, by Plancherel's theorem and the inversion formula for the Fourier transform, if $f \in \mathcal{B}$,

$$\begin{aligned} \langle f, T_x s \rangle &= \langle \hat{f}, e^{-2\pi i x \cdot \xi} \hat{s} \rangle = \int_{[-1/2, 1/2]^d} e^{2\pi i x \cdot \xi} \hat{f}(\xi) d\xi \\ &= \int e^{2\pi i x \cdot \xi} \hat{f}(\xi) d\xi = f(x). \end{aligned}$$

Since the eigenfunctions ϕ_l form an orthonormal basis for \mathcal{B} , the factor $m(x_j)$ in (14) is majorized by

$$\begin{aligned} m(x_j) &= \sum_{l=1}^N |\phi_l(x_j)|^2 = \sum_{l=1}^N |\langle \phi_l, T_{x_j} s \rangle|^2 \\ &\leq \sum_{l=1}^{\infty} |\langle \phi_l, T_{x_j} s \rangle|^2 = \|T_{x_j} s\|_2^2 = 1. \end{aligned}$$

Since $T_j^2 \leq T_j$ and the expectation preserves the cone of positive (semi)definite matrices (see, e.g., [22]), we have $\mathbb{E}(T_j^2) \leq \mathbb{E}(T_j) = R^{-d}\Delta$, and

$$\mathbb{E}(X_j^2) = \mathbb{E}(T_j^2) - R^{-2d}\Delta^2 \leq R^{-d}\Delta.$$

(iii) Now the variance of the sum of positive (semi)definite random matrices is majorized by

$$\sigma^2 = \left\| \sum_{j=1}^r \mathbb{E}(X_j^2) \right\| \leq \left\| \sum_{j=1}^r \mathbb{E}(T_j) \right\| = \frac{r}{R^d} \|\Delta\| \leq \frac{r}{R^d}. \quad \square$$

End of the proof of Proposition 2. Now we have all information to finish the proof of Proposition 2. Since $\lambda_{\min}(T) = -\lambda_{\max}(-T)$, we substitute these estimates into the matrix Bernstein inequality with $t = r\nu/R^d$, and obtain that

$$\mathbb{E} \left(\lambda_{\min} \left(\sum_{j=1}^r (T_j - \mathbb{E}(T_j)) \right) \leq -r\nu/R^d \right) \leq N \exp \left(-\frac{r^2 \nu^2 R^{-2d}/2}{rR^{-d} + r\nu R^{-d}/3} \right).$$

Combined with (11), the proposition is proved. \square

Random matrix theory offers several methods to obtain probability estimates for the spectrum of random matrices. In [2], we used the entropy method. We also mention the influential work of Rudelson [15] and the recent papers [11], [16] on random matrices with independent columns. The matrix Bernstein inequality offers a new approach and makes the probabilistic part of the argument almost painless. The matrix Bernstein inequality was first derived in [1] and improved in several subsequent papers, in particular in [13]. The version with the best constants is due to Tropp [22]. Matrix Bernstein inequalities also simplify many probabilistic arguments in compressed sensing; see [4] and [23].

3. From sampling of prolate spheroidal functions to relevant sampling of bandlimited functions

Let α be the value of the N th eigenvalue of A_R , that is, $\alpha = \lambda_N$, let $E = E_N$ be the orthogonal projections from \mathcal{B} onto \mathcal{P}_N , and let $F = F_N = I - E_N$. Intuitively, since $f \in \mathcal{B}(R, \delta)$ is essentially supported on the cube C_R , it should be close to the span of the largest eigenfunctions of A_R and thus Ff should be small. The following lemma gives a precise estimate. Compare also with the proof of [9, Theorem 3].

LEMMA 5. *If $f \in \mathcal{B}(R, \delta)$, then*

$$\begin{aligned} \|Ef\|_2^2 &\geq \left(1 - \frac{\delta}{1 - \alpha}\right) \|f\|_2^2, \\ \|Ef\|_{2,R}^2 &\geq \alpha \left(1 - \frac{\delta}{1 - \alpha}\right) \|f\|_2^2, \\ \|Ff\|_2^2 &\leq \frac{\delta}{1 - \alpha} \|f\|_2^2. \end{aligned}$$

Proof. Expand $f \in \mathcal{B}$ with respect to the prolate spheroidal functions as $f = \sum_{j=1}^{\infty} c_j \phi_j$. Without loss of generality, we may assume that $\|f\|_2 = \|c\|_2 = 1$. Since $f \in \mathcal{B}(R, \delta)$, we have that

$$1 - \delta \leq \|f\|_{2,R}^2 = \int_{C_R} |f(t)|^2 dt = \langle A_R f, f \rangle = \sum_{j=1}^{\infty} |c_j|^2 \lambda_j.$$

Set

$$A = \|Ef\|_2^2 = \sum_{j=1}^N |c_j|^2$$

and $B = \sum_{j>N} |c_j|^2 = 1 - A = \|Ff\|_2^2$. Since $\lambda_j \leq \lambda_N = \alpha$ for $j > N$, we estimate $A = \|Ef\|_2^2$ as follows:

$$\begin{aligned} A &= \sum_{j=1}^N |c_j|^2 \geq \sum_{j=1}^N |c_j|^2 \lambda_j \\ &= \sum_{j=1}^{\infty} |c_j|^2 \lambda_j - \sum_{j=N+1}^{\infty} |c_j|^2 \lambda_j \\ &\geq 1 - \delta - \lambda_N \sum_{j=N+1}^{\infty} |c_j|^2 \\ &= 1 - \delta - \alpha(1 - A). \end{aligned}$$

The inequality $A \geq 1 - \delta - \alpha(1 - A)$ implies that $\|Ef\|_2^2 = A \geq 1 - \frac{\delta}{1-\alpha}$ and using the orthogonal decomposition $f = Ef + Ff$,

$$B = \|Ff\|_2^2 \leq \frac{\delta}{1-\alpha}.$$

Finally, $\|Ef\|_{2,R}^2 = \sum_{j=1}^N \lambda_j |c_j|^2 \geq \alpha A \geq \alpha(1 - \frac{\delta}{1-\alpha})$, as claimed. \square

REMARK (due to J.-L. Romero). As mentioned in [2], if $f \in \mathcal{B}(R, \delta)$ and $f(x_j) = 0$ for sufficiently many samples $x_j \in C_R$, then $f \equiv 0$. However, f cannot be completely determined by samples in C_R alone. This is a consequence of the fact that $\mathcal{B}(R, \delta)$ is not a linear space. Given a finite subset $S \subseteq C_R$, consider the finite-dimensional subspace \mathcal{H}_0 of \mathcal{B} spanned by the reproducing kernels $T_x s$, $x \in S$. If $\phi \in \mathcal{H}_0^\perp$, then $\phi(x) = \langle \phi, T_x s \rangle = 0$ for $x \in S$. Thus by adding a function in \mathcal{H}_0^\perp of sufficiently small norm to $f \in \mathcal{B}(R, \delta)$, one obtains a different function with the same samples. More precisely, let $f \in \mathcal{B}(R, \delta)$ with $\|f\|_2 = 1$ and $\int_{C_R} |f(x)|^2 dx = \gamma > 1 - \delta$ and $\phi \in \mathcal{H}_0^\perp$ with $\|\phi\|_2 = 1$. Then $f(x) + \varepsilon\phi(x) = f(x)$ for $x \in S$ and $f + \varepsilon\phi \in \mathcal{B}(R, \delta)$ for sufficiently small $\varepsilon > 0$.

Despite this non-uniqueness, one can approximate f from the samples up to an accuracy δ , as is shown by the next lemma.

We will require a standard estimate for sampled 2-norms, a so-called Plancherel–Polya–Nikolskij inequality [21]. Assume that $\mathcal{X} = \{x_j\} \subseteq \mathbb{R}^d$ is relatively separated, that is, the “covering index”

$$\max_{k \in \mathbb{Z}^d} \#\mathcal{X} \cap (k + [-1/2, 1/2]^d) =: N_0 < \infty$$

is finite. Then there exists a constant $\kappa > 0$, such that

$$(15) \quad \sum_{j=1}^{\infty} |f(x_j)|^2 \leq \kappa N_0 \|f\|_2^2 \quad \text{for all } f \in \mathcal{B}.$$

The constant κ can be chosen as $\kappa = e^{d\pi}$. Since the standard proof in [21] uses a maximal inequality with an non-explicit constant, we will give a simple argument using Taylor series in the [Appendix](#).

LEMMA 6. *Let $\{x_j : j = 1, \dots, r\}$ be a finite subset of C_R with covering index N_0 . Then the solution to the least square problem*

$$(16) \quad p_{\text{opt}} = \arg \min_{p \in \mathcal{P}_N} \left\{ \sum_{j=1}^r |f(x_j) - p(x_j)|^2 \right\}$$

satisfies the error estimate

$$(17) \quad \sum_{j=1}^r |f(x_j) - p_{\text{opt}}(x_j)|^2 \leq N_0 \kappa \frac{\delta}{1-\alpha} \|f\|_2^2 \quad \text{for all } f \in \mathcal{B}(R, \delta).$$

Proof. We combine Lemma 5 with (15).

$$\begin{aligned} \sum_{j=1}^r |f(x_j) - p_{\text{opt}}(x_j)|^2 &\leq \sum_{j=1}^r |f(x_j) - Ef(x_j)|^2 \\ &= \sum_{j=1}^r |Ff(x_j)|^2 \leq \kappa N_0 \|Ff\|_2^2 \\ &\leq \kappa N_0 \frac{\delta}{1-\alpha} \|f\|_2^2. \end{aligned} \quad \square$$

Next, we compare sampling inequalities for the space of prolate polynomials \mathcal{P}_N to sampling inequalities for functions in $\mathcal{B}(R, \delta)$.

LEMMA 7. *Let $\{x_j : j = 1, \dots, r\}$ be a finite subset of C_R with covering index N_0 .*

If the inequality

$$(18) \quad \frac{1}{r} \sum_{j=1}^r (|p(x_j)|^2 - R^{-d} \|p\|_{2,R}^2) \geq -\frac{\nu}{R^d} \|p\|_2^2$$

holds for all $p \in \mathcal{P}_N$, then the inequality

$$(19) \quad \sum_{j=1}^r |f(x_j)|^2 \geq A \|f\|_2^2$$

holds for all $f \in \mathcal{B}(R, \delta)$ with a constant

$$A = \frac{r}{R^d} \left(\alpha - \frac{\alpha\delta}{1-\alpha} - \nu \right) - 2\kappa N_0 \frac{\delta}{1-\alpha}.$$

REMARK. For A to be positive, we need

$$r \geq R^d \frac{2\kappa N_0 \frac{\delta}{1-\alpha}}{\alpha - \frac{\alpha\delta}{1-\alpha} - \nu}.$$

Proof of Lemma 7. Using the triangle inequality and the orthogonal decomposition $f = Ef + Ff$, we estimate

$$\left(\sum_{j=1}^r |f(x_j)|^2 \right)^{1/2} \geq \left(\sum_{j=1}^r |Ef(x_j)|^2 \right)^{1/2} - \left(\sum_{j=1}^r |Ff(x_j)|^2 \right)^{1/2}.$$

Taking squares and using (15) on Ef and Ff in the cross product term, we continue as

$$\begin{aligned} \sum_{j=1}^r |f(x_j)|^2 &\geq \sum_{j=1}^r |Ef(x_j)|^2 - 2 \left(\sum_{j=1}^r |Ef(x_j)|^2 \right)^{1/2} \left(\sum_{j=1}^r |Ff(x_j)|^2 \right)^{1/2} \\ &\quad + \sum_{j=1}^r |Ff(x_j)|^2 \\ &\geq \sum_{j=1}^r |Ef(x_j)|^2 - 2\kappa N_0 \|Ef\|_2 \|Ff\|_2 \\ &\geq \sum_{j=1}^r |Ef(x_j)|^2 - 2\kappa N_0 \frac{\delta}{1-\alpha} \|f\|_2^2, \end{aligned}$$

since by Lemma 5, $\|Ff\|_2^2 \leq \frac{\delta}{1-\alpha} \|f\|_2^2$ and $\|Ef\|_2 \leq \|f\|_2$. Now we make use of hypothesis (18) and Lemma 5 and obtain

$$\begin{aligned} \sum_{j=1}^r |f(x_j)|^2 &\geq \sum_{j=1}^r |Ef(x_j)|^2 - 2\kappa N_0 \frac{\delta}{1-\alpha} \|f\|_2^2 \\ &\geq \frac{r}{R^d} \|Ef\|_{2,R}^2 - \frac{\nu r}{R^d} \|Ef\|_2^2 - 2\kappa N_0 \frac{\delta}{1-\alpha} \|f\|_2^2 \\ &\geq \frac{\alpha r}{R^d} \left(1 - \frac{\delta}{1-\alpha} \right) \|f\|_2^2 - \frac{\nu r}{R^d} \|f\|_2^2 - 2\kappa N_0 \frac{\delta}{1-\alpha} \|f\|_2^2. \end{aligned}$$

So we may choose A to be

$$A = \frac{r}{R^d} \left(\alpha - \frac{\alpha\delta}{1-\alpha} - \nu \right) - 2\kappa N_0 \frac{\delta}{1-\alpha}. \quad \square$$

The final ingredient we need is a deviation inequality for the covering index $N_0 = \max_{k \in \mathbb{Z}^d} \{x_j\} \cap (k + [-1/2, 1/2]^d)$.

LEMMA 8. *Suppose $R \geq 2$ and $\{x_j : j = 1, \dots, r\}$ are independent and identically distributed random variables that are uniformly distributed over C_R . Let $a > R^{-d}$. Then*

$$\mathbb{P}(N_0 > ar) \leq (R+2)^d \exp(-r(a \log(aR^d) - (a - R^{-d}))).$$

Proof. Let $D_k = k + [-1/2, 1/2]^d$ for $k \in \mathbb{Z}^d$. Note that we need at most $(R+2)^d$ of the D_k 's to cover C_R . If $N_0 > ar$, then for at least one k , D_k must contain at least ar of the x_j 's. Therefore,

$$(20) \quad \mathbb{P}(N_0 > ar) \leq (R+2)^d \max_{k \in \mathbb{Z}^d} \mathbb{P}(\#\{x_j\} \cap D_k > ar).$$

Fix $k \in \mathbb{Z}^d$. For any $b > 0$, by Chebyshev's inequality

$$\begin{aligned} \mathbb{P}(\#\{x_j\} \cap D_k > ar) &= \mathbb{P}\left(\sum_{j=1}^r \chi_{D_k}(x_j) > ar\right) \\ &= \mathbb{P}\left(\exp\left(b \sum_{j=1}^r \chi_{D_k}(x_j)\right) > e^{bar}\right) \\ &\leq e^{-bar} \mathbb{E} \exp\left(b \sum_{j=1}^r \chi_{D_k}(x_j)\right). \end{aligned}$$

Since the x_j are uniformly distributed over C_R , then $\chi_{D_k}(x_j)$ is equal to 1 with probability at most R^{-d} and otherwise equals zero. Therefore, using the independence,

$$\begin{aligned} \mathbb{P}(\#\{x_j\} \cap D_k > ar) &\leq e^{-bar} \prod_{j=1}^r \mathbb{E} e^{b\chi_{D_k}(x_j)} \\ &\leq e^{-bar} ((1 - R^{-d}) + e^b R^{-d})^r \\ &= e^{-bar} ((1 + (e^b - 1)R^{-d})^r \\ &\leq e^{-bar} (\exp((e^b - 1)R^{-d}))^r. \end{aligned}$$

With the optimal choice $b = \log(aR^d)$ the last term is then

$$\exp\left(-r(a \log(aR^d) - (a - R^{-d}))\right).$$

Substituting this in (20) proves the lemma. \square

By combining the finite-dimensional result of Proposition 2 with the estimates of Lemmas 7 and 8 and the appropriate choice of the free parameters, we obtain the following theorem.

THEOREM 9. *Let $\{x_j : j \in \mathbb{N}\}$ be a sequence of independent and identically distributed random variables that are uniformly distributed in C_R . Suppose $R \geq 2$,*

$$\delta < \frac{1}{2(1 + 12\kappa)}$$

and

$$\nu < \frac{1}{2} - \delta(1 + 12\kappa),$$

where $\kappa = e^{d\pi}$. Let

$$(21) \quad A = \frac{r}{R^d} \left(\frac{1}{2} - \delta - \nu - 12\delta\kappa \right).$$

Then the sampling inequality

$$(22) \quad A\|f\|_2^2 \leq \sum_{j=1}^r |f(x_j)|^2 \leq r\|f\|_2^2 \quad \text{for all } f \in \mathcal{B}(R, \delta)$$

holds with probability at least

$$(23) \quad 1 - R^d \exp\left(-\frac{\nu^2 r/2}{R^d(1+\nu/3)}\right) - (R+2)^d \exp\left(-\frac{r}{R^d}(3\log 3 - 2)\right).$$

Proof. Since $|f(x)| \leq \|f\|_2$ for $f \in \mathcal{B}$, the right-hand inequality in (22) is immediate. We take $\alpha = 1/2$ and $N = R^d$ in Proposition 2 and $a = 3R^{-d}$ in Lemma 8. Let

$$V_1 = \left\{ \inf_{f \in \mathcal{P}_N, \|f\|_2=1} \frac{1}{r} \sum_{j=1}^r \left(|f(x_j)|^2 - \frac{1}{R^d} \|f\|_{2,R}^2 \right) \leq -\frac{\nu}{R^d} \right\}$$

and let

$$V_2 = \{N_0 > ar\}.$$

By Proposition 2 and Lemma 8, the probability of $(V_1 \cup V_2)^c$ is bounded below by (23). By Lemma 7,

$$\frac{1}{r} \sum_{j=1}^r |f(x_j)|^2 \geq A\|f\|_2^2$$

for all $f \in \mathcal{B}(R, \delta)$ on the set $(V_1 \cup V_2)^c$. With $\alpha = 1/2$ and $N_0 = 3R^{-d}r$ the lower bound A of Lemma 7 simplifies to $A = \frac{r}{R^d}(\frac{1}{2} - \delta - \nu - 12\delta\kappa)$. Our assumptions on δ and ν guarantee that $A > 0$. \square

The formulation of Theorem 1 now follows. With $N = R^d$ and $0 < \nu < 1/2 - \delta < 1/2$, if $\varepsilon > 0$ is given and

$$(24) \quad r \geq \max\left(R^d \frac{1+\nu/3}{\nu^2} \log \frac{2R^d}{\varepsilon}, \frac{R^d}{3\log 3 - 2} \log \frac{2(R+2)^d}{\varepsilon} \right) \\ = R^d \frac{1+\nu/3}{\nu^2} \log \frac{2R^d}{\varepsilon},$$

then the probability in (23) will be larger than $1 - \varepsilon$.

REMARK. Observe that the parameters δ and R are not independent. As mentioned in [2, p. 14], for $\mathcal{B}(R, \delta)$ to be non-empty, we need $\delta \geq 2\pi d\sqrt{2R}e^{-\pi R}$ (up to terms of higher order). Thus for small δ as in Theorem 9, we need to choose R of order $R \approx c \log(d/\delta)$.

Appendix: The Plancherel–Polya inequality

We finish by showing that the constant κ in the Plancherel–Polya inequality (15) can be chosen explicitly to be $\kappa = e^{d\pi}$. The argument is simple and well known, see, for example, [5].

LEMMA A.1. *Let $\{x_j : j \in \mathbb{N}\}$ be a set in \mathbb{R}^d with covering index N_0 . Then*

$$\sum_{j=1}^{\infty} |f(x_j)|^2 \leq N_0 e^{d\pi} \|f\|_2^2.$$

Proof. Let $k \in \mathbb{Z}^d$ and $x_j \in k + [-1/2, 1/2] =: D_k$. Then $\|x_j - k\|_{\infty} \leq 1/2$. Consider the Taylor expansion of $f(x_j)$ at k (with the usual multi-index notation):

$$|f(x_j)| = \left| \sum_{\alpha \geq 0} \frac{D^\alpha f(k)}{\alpha!} (x_j - k)^\alpha \right| \leq \sum_{\alpha \geq 0} \frac{|D^\alpha f(k)|}{\alpha!} \left(\frac{1}{2}\right)^{|\alpha|}.$$

We now let $\theta \in (0, 1)$ and apply Cauchy–Schwarz:

$$\begin{aligned} (25) \quad |f(x_j)|^2 &\leq \sum_{\alpha \geq 0} \frac{1}{\alpha!} \left(\frac{1}{2}\right)^{2\theta|\alpha|} \sum_{\alpha \geq 0} \frac{|D^\alpha f(k)|^2}{\alpha!} \left(\frac{1}{2}\right)^{2(1-\theta)|\alpha|} \\ &= e^{d/4^\theta} \sum_{\alpha \geq 0} \frac{|D^\alpha f(k)|^2}{\alpha!} \left(\frac{1}{2}\right)^{2(1-\theta)|\alpha|}. \end{aligned}$$

If $f \in \mathcal{B}$, then by Shannon’s sampling theorem (or because the reproducing kernels $T_{k,s}, k \in \mathbb{Z}^d$, form an orthonormal basis of \mathcal{B}) we have

$$\sum_{k \in \mathbb{Z}^d} |f(k)|^2 = \|f\|_2^2 \quad \forall f \in \mathcal{B}.$$

To estimate the partial derivatives we use Bernstein’s inequality $\|D^\alpha f\|_2 \leq \pi^{|\alpha|} \|f\|_2$.

We first assume that $N_0 = 1$, that is, each cube D_k contains at most one of the x_j ’s. Then we obtain, after interchanging the order of summation

$$\begin{aligned} (26) \quad \sum_{j \in \mathbb{N}} |f(x_j)|^2 &\leq e^{d/4^\theta} \sum_{\alpha \geq 0} \sum_{k \in \mathbb{Z}^d} \frac{|D^\alpha f(k)|^2}{\alpha!} \left(\frac{1}{2}\right)^{2(1-\theta)|\alpha|} \\ &= e^{d/4^\theta} \sum_{\alpha \geq 0} \left(\frac{1}{2}\right)^{2(1-\theta)|\alpha|} \frac{\|D^\alpha f\|_2^2}{\alpha!} \\ &\leq e^{d/4^\theta} \sum_{\alpha \geq 0} \left(\frac{1}{2}\right)^{2(1-\theta)|\alpha|} \frac{\pi^{2|\alpha|}}{\alpha!} \|f\|_2^2 = e^{d/4^\theta} e^{d\pi^2/4^{1-\theta}} \|f\|_2^2. \end{aligned}$$

The choice $4^\theta = 2/\pi$ yields the constant $\kappa = e^{d/4^\theta} e^{d\pi^2/4^{1-\theta}} = e^{d\pi}$. For arbitrary N_0 we obtain

$$\sum_{j \in \mathbb{N}} |f(x_j)|^2 = \sum_{k \in \mathbb{Z}^d} \sum_{\{j: x_j \in D_k\}} |f(x_j)|^2 \leq N_0 e^{d\pi} \|f\|_2^2,$$

as claimed. \square

Possibly the Plancherel–Polya inequality could be improved to a local estimate of the form $\sum_{x_j \in C_R} |f(x_j)|^2 \leq \tilde{\kappa} N_0 \|f\|_{2,R}^2$, but we did not pursue this question.

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RICHARD F. BASS, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CT 06269-3009, USA

E-mail address: r.bass@uconn.edu

KARLHEINZ GRÖCHENIG, FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, OSKAR-MORGENSTERN-PLATZ 1, 1090 VIEN, AUSTRIA

E-mail address: karlheinz.groechenig@univie.ac.at