Large Deviations for Riesz Potentials of Additive Processess

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May 13, 2008

Abstract

We study functionals of the form

$$\zeta_t = \int_0^t \cdots \int_0^t |X_1(s_1) + \cdots + X_p(s_p)|^{-\sigma} ds_1 \cdots ds_p,$$

where $X_1(t), \cdots, X_p(t)$ are i.i.d. d-dimensional symmetric stable processes of index $0 < \beta \le 2$. We obtain results about the large deviations and laws of the iterated logarithm for ζ_t.

Résumé

Nous étudions les fonctionelles de la forme

$$\zeta_t = \int_0^t \cdots \int_0^t |X_1(s_1) + \cdots + X_p(s_p)|^{-\sigma} ds_1 \cdots ds_p,$$

où $X_1(t), \ldots, X_p(t)$ sont des processus stables symétriques indépendants et identiquement distribués d'ordre $0 < \beta \le 2$. Nous obtenons des résultats sur les grandes déviations et les lois du logarithme itéré.

*Research partially supported by NSF grant #DMS-0601783
†Research partially supported by NSF grant #DMS-0704024.
‡Research partially supported by grants from the NSF and from PSC-CUNY.

1 Introduction

Let $X_1(t), \cdots, X_p(t)$ be i.i.d. d-dimensional symmetric stable process of index $0 < \beta \leq 2$. We use the notation $X(t)$ for a stable process with the same distribution as $X_1(t), \cdots X_p(t)$. Thus

$$\mathbb{E}e^{i\lambda \cdot X_t} = e^{-t|\lambda|^\beta}, \quad t \geq 0, \quad \lambda \in \mathbb{R}^d. \tag{1.1}$$

In this paper we study

$$\zeta([0,t_1]\times\cdots\times[0,t_p]) = \int_0^{t_1}\cdots\int_0^{t_p} |X_1(s_1)+\cdots+X_p(s_p)|^{-\sigma} ds_1\cdots ds_p \tag{1.2}$$

and more generally

$$\zeta^z([0,t_1]\times\cdots\times[0,t_p]) = \int_0^{t_1}\cdots\int_0^{t_p} |X_1(s_1)+\cdots+X_p(s_p)-z|^{-\sigma} ds_1\cdots ds_p \tag{1.3}$$

for $z \in \mathbb{R}^d$. We show below that $\zeta^z([0,t_1]\times\cdots\times[0,t_p])$ is finite almost surely if

$$0 < \sigma < \min\{p\beta, d\}. \tag{1.4}$$

The random field $\bar{X}(t_1,\ldots,t_p) = X_1(t_1)+\cdots+X_p(t_p)$ is known as an additive process, and its occupation measure μ_A for $A \in \mathbb{R}^p_+$ is the measure on $\mathbb{R}^d$ defined by

$$\mu_A(B) = \int_A 1_{\{X_1(s_1)+\cdots+X_p(s_p)\in B\}}\, ds_1\cdots\, ds_p. \tag{1.5}$$

With this notation we have

$$\zeta^z([0,t_1]\times\cdots\times[0,t_p]) = \int_{\mathbb{R}^d} \frac{1}{|x-z|^{-\sigma}}\ \mu_{[0,t_1]\times\cdots\times[0,t_p]}(\,dx) \tag{1.6}$$

so that $\zeta^z([0,t_1]\times\cdots\times[0,t_p])$ is the Riesz potential of the occupation measure $\mu_{[0,t_1]\times\cdots\times[0,t_p]}$. (In the terminology of [5], $\zeta^z([0,t_1]\times\cdots\times[0,t_p])$ is the Riesz-Frostman potential of the occupation measure.)

Because they locally resemble stable sheets, but are more amenable to analysis, additive stable processes first arose to simplify the study of stable sheets (see Dalang and Walsh [7, 8], Kahane [11] and Kendall [12]). They also arise in the theory of intersections and self intersections of stable processes (see Le Gall, Rosen and Shieh [18], Fitzsimmons and Salisbury [9], Khoshnevisan and Xiao [15]). In addition, the study of additive processes has connections with probabilistic potential theory. We refer the reader to Hirsch and Song [10], Khoshnevisan [13], Khoshnevisan and Shi [14], Khoshnevisan and Xiao [15] for detailed discussion and further references. The present paper is a direct outgrowth of [2].

We are interested in Riesz potentials for two reasons. First of all, they provide an opportunity to study functionals of the paths which are quite singular. When $d > 1$, local times do not exist, and the Riesz potentials are an interesting substitute as an object of study. The second reason involves generalizations of the polaron problem. Donsker and Varadhan [6] show that for Brownian motion in $\mathbb{R}^3$

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\left\{ \frac{1}{t} \int_0^t \int_0^t \frac{1}{|X_s - X_r|}\, dr\, ds \right\} \\ &= \sup_{g\in\mathcal{F}_2} \left\{ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{g^2(x)g^2(y)}{|x-y|} dx\, dy - \frac{1}{2} \|\nabla f\|_2^2 \right\}. \end{aligned} \tag{1.7}$$

The object in the exponential involves a Riesz potential but here we have a single process as opposed to several independent processes.

Theorem 1.1 *Under (1.4), $\zeta^z([0,t_1] \times \cdots \times [0,t_p])$ is jointly continuous in $z, t_1, \ldots, t_p$, almost surely.*

We note for later reference that by scaling we have

$$\zeta^z([0,t]^p) \stackrel{d}{=} t^{\frac{p\beta-\sigma}{\beta}} \zeta^{z/t^{1/\beta}}([0,1]^p). \tag{1.8}$$

For $0 < \sigma < d$ let

$$\varphi_{d-\sigma}(\lambda) = \frac{C_{d,\sigma}}{|\lambda|^{d-\sigma}} \tag{1.9}$$

where $C_{d,\sigma} = \pi^{-d/2} 2^{-\sigma} \Gamma(\frac{d-\sigma}{2})/\Gamma(\frac{\sigma}{2})$. Write

$$\rho = \sup_{||f||_2=1} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{f(\lambda+\gamma) f(\gamma)}{\sqrt{1+\psi(\lambda+\gamma)}\sqrt{1+\psi(\gamma)}} d\gamma \right]^p \varphi_{d-\sigma}(\lambda) d\lambda \tag{1.10}$$

where $\psi(\lambda) = |\lambda|^\beta$ is the characteristic exponent of the stable processes. Clearly, $\rho > 0$. We will prove below that $\rho < \infty$ under condition (1.4).

Our main theorem is the large deviation principle for $\zeta\big([0,t]^p\big)$. By the scaling property (1.8) we need only consider $\zeta\big([0,1]^p\big)$ in the following theorem.

Theorem 1.2 *Under (1.4),*

$$\lim_{t\to\infty} t^{-\beta/\sigma} \log \mathbb{P}\Big\{\zeta([0,1]^p) \geq t\Big\} = -\frac{\sigma}{\beta}\Big(\frac{p\beta-\sigma}{p\beta}\Big)^{\frac{p\beta-\sigma}{\sigma}} \rho^{-\beta/\sigma} \tag{1.11}$$

where ρ is given in (1.10).

The next Theorem treats the large deviations of

$$\zeta^*\big([0,1]^p\big) =: \sup_{z\in\mathbb{R}^d} \zeta^z\big([0,1]^p\big).$$

Theorem 1.3 *Under (1.4), when $\beta = 2$*

$$\lim_{t\to\infty} t^{-\beta/\sigma} \log \mathbb{P}\Big\{\zeta^*([0,1]^p) \geq t\Big\} = -\frac{\sigma}{\beta}\Big(\frac{p\beta-\sigma}{p\beta}\Big)^{\frac{p\beta-\sigma}{\sigma}} \rho^{-\beta/\sigma} \tag{1.12}$$

while for $\beta < 2$, for some $0 < C_1 < \infty$

$$\limsup_{t\to\infty} t^{-\beta/\sigma} \log \mathbb{P}\Big\{\zeta^*([0,1]^p) \geq t\Big\} \leq -C_1 \tag{1.13}$$

where ρ is given in (1.10).

We can also find a law of the iterated logarithm for $\zeta^z([0,t]^p)$ and $\zeta^*([0,t]^p)$.

Theorem 1.4 *Under (1.4),*

$$\limsup_{t\to\infty} t^{-\frac{p\beta-\sigma}{\beta}} (\log\log t)^{-\sigma/\beta} \zeta([0,t]^p) = \Big(\frac{\sigma}{\beta}\Big)^{-\sigma/\beta} \Big(\frac{p\beta-\sigma}{p\beta}\Big)^{\frac{\sigma-p\beta}{\beta}} \rho \tag{1.14}$$

almost surely and when $\beta = 2$

$$\limsup_{t\to\infty} t^{-\frac{p\beta-\sigma}{\beta}} (\log\log t)^{-\sigma/\beta} \zeta^*([0,t]^p) = \Big(\frac{\sigma}{\beta}\Big)^{-\sigma/\beta} \Big(\frac{p\beta-\sigma}{p\beta}\Big)^{\frac{\sigma-p\beta}{\beta}} \rho. \tag{1.15}$$

In the case $\beta < 2$, there is a constant $0 < C_2 < \infty$ such that

$$\limsup_{t\to\infty} t^{-\frac{p\beta-\sigma}{\beta}}(\log\log t)^{-\sigma/\beta}\zeta^*([0,t]^p) = C_2. \tag{1.16}$$

Even when $\beta < 2$, the following lower bounds

$$\liminf_{t\to\infty} t^{-\beta/\sigma}\log\mathbb{P}\Big\{\zeta^*([0,1]^p) \geq t\Big\} \geq -\frac{\sigma}{\beta}\Big(\frac{p\beta-\sigma}{p\beta}\Big)^{\frac{p\beta-\sigma}{\sigma}}\rho^{-\beta/\sigma}$$

and

$$\limsup_{t\to\infty} t^{-\frac{p\beta-\sigma}{\beta}}(\log\log t)^{-\sigma/\beta}\zeta^*([0,t]^p) \geq \Big(\frac{\sigma}{\beta}\Big)^{-\sigma/\beta}\Big(\frac{p\beta-\sigma}{p\beta}\Big)^{\frac{\sigma-p\beta}{\beta}}\rho$$

follow trivially from (1.11) and (1.15), respectively. We believe that (1.12) and (1.15) hold for all β. In other words, we believe that the constants C_1 in (1.13) and C_2 in (1.16) are equal to

$$\frac{\sigma}{\beta}\Big(\frac{p\beta-\sigma}{p\beta}\Big)^{\frac{p\beta-\sigma}{\sigma}}\rho^{-\beta/\sigma} \quad\text{and}\quad \Big(\frac{\sigma}{\beta}\Big)^{-\sigma/\beta}\Big(\frac{p\beta-\sigma}{p\beta}\Big)^{\frac{\sigma-p\beta}{\beta}}\rho,$$

respectively.

We can obtain a variational expression for ρ. Let $\beta \leq 2$ and set

$$\mathcal{E}_\beta(f,f) =: (2\pi)^{-d}\int_{\mathbb{R}^d}|\lambda|^\beta|\widehat{f}(\lambda)|^2\,d\lambda. \tag{1.17}$$

Let

$$\mathcal{F}_\beta = \{f \in L^2(R^d)\,|\,\|f\|_2 = 1\,,\ \mathcal{E}_\beta(f,f) < \infty\}. \tag{1.18}$$

We show below that under condition (1.4)

$$\Lambda_\sigma =: \sup_{g\in\mathcal{F}_\beta}\left\{\left(\int_{(\mathbb{R}^d)^p}\frac{\prod_{j=1}^p g^2(x_j)}{|x_1+\cdots+x_p|^\sigma}\prod_{j=1}^p dx_j\right)^{1/p} - \mathcal{E}_\beta(g,g)\right\} < \infty. \tag{1.19}$$

Theorem 1.5 *Under condition (1.4)*

$$\rho = (2\pi)^{-d}(\Lambda_\sigma)^{p-(\sigma/\beta)}. \tag{1.20}$$

We now prove that $\rho < \infty$ under condition (1.4). This will follow from the next Lemma and the fact that $\beta pd/\sigma > d$ by (1.4).

Lemma 1.6 *For any f, g, h with $h \geq 0$*
(1.21)

$$\left(\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\frac{|f(\lambda+\gamma)g(\gamma)|}{\sqrt{h(\lambda+\gamma)}\sqrt{h(\gamma)}}d\gamma\right]^p\varphi_{d-\sigma}(\lambda)d\lambda\right)^{1/p} \leq C\|f\|_2\,\|g\|_2\,\|h^{-1}\|_{pd/\sigma}.$$

Proof of Lemma 1.6. By Hölder's inequality

$$\begin{aligned}(1.22)\qquad &\left[\int_{\mathbb{R}^d}\frac{|f(\lambda+\gamma)g(\gamma)|}{\sqrt{h(\lambda+\gamma)}\sqrt{h(\gamma)}}d\gamma\right]^p\\ &= \left[\int_{\mathbb{R}^d}|f(\lambda+\gamma)g(\gamma)|^{(p-1)/p}\frac{|f(\lambda+\gamma)g(\gamma)|^{1/p}}{\sqrt{h(\lambda+\gamma)}\sqrt{h(\gamma)}}d\gamma\right]^p\\ &\leq \left(\int_{\mathbb{R}^d}|f(\lambda+\gamma)g(\gamma)|d\gamma\right)^{p-1}\int_{\mathbb{R}^d}\frac{|f(\lambda+\gamma)g(\gamma)|}{\big(h(\lambda+\gamma)\big)^{p/2}\big(h(\gamma)\big)^{p/2}}d\gamma.\end{aligned}$$

By the Cauchy-Schwarz inequality and translation invariance,

$$(1.23)\qquad \int_{\mathbb{R}^d}|f(\lambda+\gamma)g(\gamma)|d\gamma \leq \|f\|_2\|g\|_2.$$

Hence,

$$\begin{aligned}(1.24)\quad &\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\frac{|f(\lambda+\gamma)g(\gamma)|}{\sqrt{h(\lambda+\gamma)}\sqrt{h(\gamma)}}d\gamma\right]^p\varphi_{d-\sigma}(\lambda)\,d\lambda\\ &\leq \|f\|_2^{p-1}\|g\|_2^{p-1}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\frac{|f(\lambda+\gamma)g(\gamma)|}{\big(h(\lambda+\gamma)\big)^{p/2}\big(h(\gamma)\big)^{p/2}}d\gamma\right)\varphi_{d-\sigma}(\lambda)\,d\lambda\\ &= C_{d,\sigma}\|f\|_2^{p-1}\|g\|_2^{p-1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{F(\gamma)G(\lambda)}{|\lambda-\gamma|^{d-\sigma}}d\gamma\,d\lambda\end{aligned}$$

where

$$(1.25)\qquad F(\gamma) =: \frac{|f(\gamma)|}{\big(h(\gamma)\big)^{p/2}},\qquad G(\lambda) =: \frac{|g(\lambda)|}{\big(h(\lambda)\big)^{p/2}}.$$

Sobolev's inequality, [5, p. 275], says that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{F(\gamma)G(\lambda)}{|\lambda-\gamma|^{d-\sigma}}d\gamma\,d\lambda\le C\|F\|_r\|G\|_s \tag{1.26}$$

for any $r,s>1$ with $s^{-1}+r^{-1}=1+\sigma/d$. In particular,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{F(\gamma)G(\lambda)}{|\lambda-\gamma|^{d-\sigma}}d\gamma\,d\lambda\le C\|F\|_{2d/(d+\sigma)}\|G\|_{2d/(d+\sigma)} \tag{1.27}$$

and by Hölder's inequality

$$\begin{aligned}\int_{\mathbb{R}^d}|F(\gamma)|^{2d/(d+\sigma)}\,d\lambda &= \int_{\mathbb{R}^d}\frac{|f(\gamma)|^{2d/(d+\sigma)}}{\big(h(\gamma)\big)^{pd/(d+\sigma)}}\,d\lambda \\ &\le \||f|^{2d/(d+\sigma)}\|_{(d+\sigma)/d}\;\|h^{-pd/(d+\sigma)}\|_{(d+\sigma)/\sigma} \\ &\le \|f\|_2^{2d/(d+\sigma)}\;\|h^{-pd/\sigma}\|_1^{\sigma/(d+\sigma)}.\end{aligned} \tag{1.28}$$

Thus

$$\|F\|_{2d/(d+\sigma)}\le\|f\|_2\;\|h^{-1}\|_{pd/\sigma}^{p/2}. \tag{1.29}$$

A similar inequality holds for G and g. Our Lemma follows. □

We next show that $\zeta^z([0,t_1]\times\cdots\times[0,t_p])$ is finite almost surely under condition (1.4). Let $p_t(x)$ denote the transition density for the symmetric stable process in $\mathbb{R}^d$ of index β. As usual, we define the β-potential density by

$$u^\beta(x)=\int_0^\infty e^{-\beta t}p_t(x)\,dt. \tag{1.30}$$

By independence

$$\begin{aligned}&\mathbb{E}\left(\zeta^z([0,t_1]\times\cdots\times[0,t_p])\right)\\ &=\int_0^{t_1}\cdots\int_0^{t_p}\int_{(\mathbb{R}^d)^p}\frac{1}{|x_1+\cdots+x_p-z|^\sigma}\prod_{j=1}^p p_{s_j}(x_j)\,dx_j\,ds_j\\ &\le e^{\sum_{j=1}^p t_j}\int_{(\mathbb{R}^d)^p}\frac{1}{|x_1+\cdots+x_p-z|^\sigma}\prod_{j=1}^p\int_0^{t_p}e^{-s_j}p_{s_j}(x_j)\,ds_j\,dx_j\\ &\le e^{\sum_{j=1}^p t_j}\int_{(\mathbb{R}^d)^p}\frac{1}{|x_1+\cdots+x_p-z|^\sigma}\prod_{j=1}^p u^1(x_j)\,dx_j\\ &\le e^{\sum_{j=1}^p t_j}\int_{\mathbb{R}^d}\frac{1}{|x-z|^\sigma}(u^1*\cdot*u^1)(x)\,dx\end{aligned} \tag{1.31}$$

where $(u^1 * \cdot * u^1)$ is the p-fold convolution of u^1 with itself. $u^1(x)$ is integrable, monotone decreasing in $|x|$, and asymptotic at $x = 0$ to $u^0(x) = C|x|^{-\max(0,(d-\beta))}$. Hence $(u^1 * \cdot * u^1)$ is integrable and bounded except (possibly) at $x = 0$ where it is asymptotic to $C|x|^{-\max(0,(d-p\beta))}$. Hence (1.31) is finite if (1.4) holds. $\square$

Outline: In Section 2 we prove Theorem 1.1 and provide the general outline for our proof of the main result of this paper, Theorem 1.2, on large deviations. The details are carried out in Sections 3-6. Section 7 is devoted to the proof of the variational formula of Theorem 1.5, while in Section 8 we prove Theorem 1.3 on large deviations for ζ^*. Section 9 establishes Theorem 1.4 on laws of the iterated logarithm. Finally, an Appendix, Section 10, provides certain Sobolev-type inequalities which are needed for our proofs.

Conventions: We define

$$\widehat{f}(\lambda) = \int_{\mathbb{R}^d} e^{ix\cdot\lambda} f(x)\,dx. \tag{1.32}$$

With this notation

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix\cdot\lambda} \widehat{f}(\lambda)\,dx, \tag{1.33}$$

$$\widehat{f * g}(\lambda) = \widehat{f}(\lambda)\widehat{g}(\lambda), \qquad \widehat{fg}(\lambda) = (2\pi)^{-d}\widehat{f}(\lambda) * \widehat{g}(\lambda), \tag{1.34}$$

and Parseval's identity is

$$\langle f, g\rangle = (2\pi)^{-d}\langle \widehat{f}, \widehat{g}\rangle. \tag{1.35}$$

If $\Phi \in \mathcal{S}'(\mathbb{R}^d)$, the set of tempered distributions on $\mathbb{R}^d$, we use $\mathcal{F}(\Phi)$ to denote the Fourier transform of Φ, so that for any $f \in \mathcal{S}(\mathbb{R}^d)$

$$\mathcal{F}(\Phi)(f) = \Phi(\widehat{f}). \tag{1.36}$$

It is well known. e.g. [5, p. 156], that $\varphi_{d-\sigma} \in \mathcal{S}'(\mathbb{R}^d)$ for any $0 < \sigma < d$ and

$$\mathcal{F}(\varphi_{d-\sigma}) = \frac{1}{|x|^\sigma}. \tag{1.37}$$

2 Killing at exponential times

We begin by citing [17, Lemma 2.3].

Lemma 2.1 *Let Y be any non-negative random variable and let $\theta > 0$ be fixed. Assume that*

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^\theta}\mathbb{E}Y^n = -\kappa \tag{2.1}$$

for some $\kappa \in \mathbb{R}$. Then we have

$$\lim_{t\to\infty} t^{-1/\theta}\log\mathbb{P}\{Y \geq t\} = -\theta e^{\kappa/\theta}. \tag{2.2}$$

In [17], König and Mörters assume that θ is a positive integer. By examining their proof, we find that θ can be any positive number.

Using this Lemma, Theorem 1.2 will follow from

$$\lim_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^{\sigma/\beta}}\mathbb{E}\zeta([0,1]^p)^m = \log\Big(\frac{p\beta}{p\beta-\sigma}\Big)^{\frac{p\beta-\sigma}{\beta}} + \log\rho. \tag{2.3}$$

In this section we show that (2.3) follows from

$$\lim_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^p}\mathbb{E}\zeta([0,\tau_1]\times\cdots\times[0,\tau_p])^m = \log\rho \tag{2.4}$$

where $\tau_1,\cdots,\tau_p$ are i.i.d. exponential times with parameter 1 independent of X.

In the rest of the paper, we use $\tau_1,\cdots,\tau_p$ to represent independent exponential times with mean 1, and we use Σ_n for the set of all permutations on $\{1,\cdots,n\}$. We assume that $\{\tau_1,\cdots,\tau_p\}$ and $\{X_1(t),\cdots,X_p(t)\}$ are independent. We begin with a useful representation of the m'th moment of the random variable

$$\zeta\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big). \tag{2.5}$$

Write $\psi(\lambda) = |\lambda|^\beta$ and $Q(\lambda) = \big[1+\psi(\lambda)\big]^{-1}$.

Lemma 2.2

$$\mathbb{E}\Big[\zeta^z([0,\tau_1]\times\cdots\times[0,\tau_p])^m\Big] \tag{2.6}$$
$$= \int_{(\mathbb{R}^d)^m} e^{i\sum_{k=1}^m \lambda_k\cdot z}\Big[\sum_{\pi\in\Sigma_m}\prod_{k=1}^m Q\Big(\sum_{j=1}^k \lambda_{\pi(j)}\Big)\Big]^p \prod_{k=1}^m \varphi_{d-\sigma}(\lambda_k)\,d\lambda_k$$

and for any fixed $t_1,\ldots,t_p>0$

$$\mathbb{E}\Big[\zeta([0,t_1]\times\cdots\times[0,t_p])^n\Big] \le (t_1\cdots t_p)^{\frac{\beta p-\sigma}{\beta p}n}\mathbb{E}\Big[\zeta\big([0,1]^p\big)^n\Big]. \tag{2.7}$$

The proof of Lemma 2.2 is given in Section 4.

Proof of Theorem 1.1 : Using the multi-parameter version of Kolmogorov's Lemma it suffices to show that we can find $\delta>0$ such that for all n and M we can find a $C<\infty$ such that

$$\mathbb{E}\Big[\Big|\zeta^z([0,t_1]\times\cdots\times[0,t_p]) - \zeta^{z'}([0,t_1']\times\cdots\times[0,t_p'])\Big|^n\Big] \tag{2.8}$$
$$\le C|(z,t_1,\ldots,t_p)-(z',t_1',\ldots,t_p')|^{\delta n}$$

uniformly in $(z,t_1,\cdots,t_p),(t_1,\cdots,t_p,z')\in\mathbb{R}^d\times[0,M]^p$. To this end it suffices to show separately that

$$\mathbb{E}\Big[\Big|\zeta^z([0,t_1]\times\cdots\times[0,t_p]) - \zeta^{z'}([0,t_1]\times\cdots\times[0,t_p])\Big|^n\Big] \tag{2.9}$$
$$\le C|z-z'|^{\delta n}$$

uniformly in $z,z'\in\mathbb{R}^d,(t_1,\ldots,t_p)\in[0,M]^p$ and

$$\mathbb{E}\Big[\Big|\zeta^z([0,t_1]\times\cdots\times[0,t_p]) - \zeta^{z}([0,t_1']\times\cdots\times[0,t_p'])\Big|^n\Big] \tag{2.10}$$
$$\le C|(t_1,\ldots,t_p)-(t_1',\ldots,t_p')|^{\delta n}$$

uniformly in $z\in\mathbb{R}^d,(t_1,\ldots,t_p),(t_1',\ldots,t_p')\in[0,M]^p$.

For (2.9) we note first that by the Mean Value Theorem, for any $u,v\ge0$ we have $|u^{-\sigma}-v^{-\sigma}|\le\sigma|u-v|\max(u^{-\sigma-1},v^{-\sigma-1})$. Applying this to $u=|x-z|$, $v=|x-z'|$ we obtain

$$\Big||x-z|^{-\sigma}-|x-z'|^{-\sigma}\Big|\le C|z-z'|\left(|x-z|^{-\sigma-1}+|x-z'|^{-\sigma-1}\right). \tag{2.11}$$

Interpolating this with the obvious bound

$$\Big||x-z|^{-\sigma}-|x-z'|^{-\sigma}\Big| \le \left(|x-z|^{-\sigma}+|x-z'|^{-\sigma}\right) \tag{2.12}$$

we see that for any $0\le\delta\le 1$

$$\Big||x-z|^{-\sigma}-|x-z'|^{-\sigma}\Big| \le C|z-z'|^{\delta}\left(|x-z|^{-\sigma-\delta}+|x-z'|^{-\sigma-\delta}\right). \tag{2.13}$$

Then writing

$$\zeta^z_\sigma([0,t_1]\times\cdots\times[0,t_p]) = \int_0^{t_1}\cdots\int_0^{t_p} |X_1(s_1)+\cdots+X_p(s_p)-z|^{-\sigma} ds_1\cdots ds_p \tag{2.14}$$

and setting $\sigma'=\sigma+\delta$ for $\delta>0$ sufficiently small so that σ' satisfies (1.4) we see that

$$\begin{aligned} &\mathbb{E}\Big[\Big|\zeta^z_\sigma([0,t_1]\times\cdots\times[0,t_p])-\zeta^{z'}_\sigma([0,t_1]\times\cdots\times[0,t_p])\Big|^n\Big] \qquad (2.15)\\ &\le C^n|z-z'|^{\delta n}\sup_z \mathbb{E}\Big[\zeta^z_{\sigma'}([0,t_1]\times\cdots\times[0,t_p])^n\Big]\\ &\le C^n e^{pM}|z-z'|^{\delta n}\sup_z \mathbb{E}\Big[\zeta^z_{\sigma'}([0,\tau_1]\times\cdots\times[0,\tau_p])^n\Big]\\ &\le C^n e^{pM}|z-z'|^{\delta n}\int_{(\mathbb{R}^d)^m}\Big[\sum_{\pi\in\Sigma_m}\prod_{k=1}^m Q\Big(\sum_{j=1}^k\lambda_{\pi(j)}\Big)\Big]^p\prod_{k=1}^m\varphi_{d-\sigma'}(\lambda_k)\,d\lambda_k \end{aligned}$$

where the last step used (2.6). By Jensen's inequality,

$$\begin{aligned} &\int_{(\mathbb{R}^d)^n}\Big[\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n Q\Big(\sum_{j=1}^k\lambda_{\sigma(j)}\Big)\Big]^p\prod_{i=1}^n\varphi_{d-\sigma'}(\lambda_i)d\lambda_i \qquad (2.16)\\ &\le (n!)^{p-1}\sum_{\sigma\in\Sigma_n}\int_{(\mathbb{R}^d)^n}\prod_{k=1}^n Q^p\Big(\sum_{j=1}^k\lambda_{\sigma(j)}\Big)\varphi_{d-\sigma'}(\lambda_i)d\lambda_i\\ &= (n!)^p\int_{(\mathbb{R}^d)^n}\prod_{k=1}^n Q^p(\lambda_k)\varphi_{d-\sigma'}(\lambda_i-\lambda_{i-1})d\lambda_i\\ &\le (n!)^p\Big(\int_{\mathbb{R}^d}\varphi_{d-\sigma'}(\lambda)Q^p(\lambda)d\lambda\Big)^n \end{aligned}$$

where the second step follows from variable substitution and the last inequality used the fact that for any two positive spherically symmetric and decreasing functions f, g

$$f * g\,(\lambda) \text{ is spherically symmetric and decreasing.} \tag{2.17}$$

(The spherical symmetry is easy. To show that $f * g\,(\lambda)$ is decreasing it suffices to prove this for simple functions, and then for indicator functions of balls centered at the origin in which case it is obvious.) Finally, the last line of (2.16) is bounded if σ' satisfies (1.4). This completes the proof of (2.9).

For (2.10) we note first that it suffices to prove a similar bound in which we vary only one of the t_j. For definiteness we vary t_1. By Hölder's inequality, for any positive function f and any conjugate r, r'

$$\int_A f(s_1, \ldots, s_p)\, ds_1 \ldots\, ds_p \le |A|^{1/r'} \left(\int_A f^r(s_1, \ldots, s_p)\, ds_1 \ldots\, ds_p \right)^{1/r} \tag{2.18}$$

where $|A|$ denotes the Lebesgue measure of $A \subseteq (\mathbb{R}_+)^p$. Hence with $t_1 > t_1'$

$$\begin{aligned} &|\zeta^z([0,t_1] \times \cdots \times [0,t_p]) - \zeta^z([0,t_1'] \times [0,t_2] \times \cdots \times [0,t_p])| \\ &\quad = \zeta^z([t_1', t_1] \times [0,t_2] \times \cdots \times [0,t_p]) \\ &\quad \le M|t_1 - t_1'|^{1/r'} \left(\int_0^{t_1} \cdots \int_0^{t_p} |X_1(s_1) + \cdots + X_p(s_p)|^{-r\sigma} ds_1 \cdots ds_p \right)^{1/r}. \end{aligned} \tag{2.19}$$

Choose a rational $r > 1$ so that $r\sigma$ satisfies (1.4). Then we can find arbitrarily large n so that n/r is an integer. For such n we can obtain (2.10) as above, and this is enough for Kolmogorov's Lemma. (In fact, using Hölder's inequality we can then obtain (2.10) for all n.) □

We state (2.4) as a theorem. The proof is given in Sections 3–6.

Theorem 2.3 *Under (1.4),*

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{(n!)^p} \mathbb{E}\Big[\zeta([0,\tau_1] \times \cdots \times [0,\tau_p])^n\Big] = \log \rho \tag{2.20}$$

where $\rho > 0$ *is given in (1.10).*

The hard part of Theorem 2.3 is the upper bound. However, it is easy to obtain a rough upper bound using (2.16). Since we will need this in the proof of Theorem 2.3 we state this rough upper bound as a lemma.

Lemma 2.4

$$\limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^p}\int_{(\mathbb{R}^d)^n}\Big[\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n Q\Big(\sum_{j=1}^k \lambda_{\sigma(j)}\Big)\Big]^p\prod_{i=1}^n\varphi_{d-\sigma}(\lambda_i)d\lambda_i \le \log\bigg(\int_{\mathbb{R}^d}\varphi_{d-\sigma}(\lambda)Q^p(\lambda)d\lambda\bigg). \tag{2.21}$$

Unfortunately, by examining the argument in (2.16)-(2.17), it is not hard to see that we do not obtain the correct constant.

We now show that Theorem 1.2 follows from Theorem 2.3.

Proof of Theorem 1.2. Using (2.7)

$$\begin{aligned}&\mathbb{E}\Big[\zeta([0,\tau_1]\times\cdots\times[0,\tau_p])^n\Big]\\ &=\int_0^\infty\cdots\int_0^\infty e^{-(t_1+\cdots+t_p)}\mathbb{E}\Big[\zeta([0,t_1]\times\cdots\times[0,t_p])^n\Big]dt_1\cdots dt_p\\ &\le\mathbb{E}\Big[\zeta([0,1]^p)^n\Big]\int_0^\infty\cdots\int_0^\infty(t_1\cdots t_p)^{\frac{\beta p-\sigma}{\beta p}n}e^{-(t_1+\cdots+t_p)}dt_1\cdots dt_p\\ &=\mathbb{E}\Big[\zeta([0,1]^p)^n\Big]\Big[\Gamma\Big(\frac{\beta p-\sigma}{\beta p}n+1\Big)\Big]^p.\end{aligned}\tag{2.22}$$

By Theorem 2.3 and Stirling's formula,

$$\liminf_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^{\sigma/\beta}}\mathbb{E}\Big[\zeta([0,1]^p)^n\Big]\ge\log\Big(\frac{\beta p}{\beta p-\sigma}\Big)^{\frac{\beta p-\sigma}{\beta}}+\log\rho. \tag{2.23}$$

On the other hand, notice that $\bar\tau\equiv\min\{\tau_1,\cdots,\tau_p\}$ has an exponential distribution with the parameter p. Hence,

$$\begin{aligned}&\mathbb{E}\Big[\zeta\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\Big]^n\ge\mathbb{E}\Big[\zeta\big([0,\bar\tau]^p\big)^n\Big]=\mathbb{E}\bar\tau^{\frac{\beta p-\sigma}{\beta}n}\mathbb{E}\Big[\zeta\big([0,1]^p\big)^n\Big]\\ &=p^{-\frac{\beta p-\sigma}{\beta}n-1}\Gamma\Big(1+\frac{\beta p-\sigma}{\beta}n\Big)\mathbb{E}\Big[\zeta\big([0,1]^p\big)^n\Big]\end{aligned}\tag{2.24}$$

where the second step follows from (1.8). By Stirling's formula we have

$$\limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^{\sigma/\beta}}\mathbb{E}\Big[\zeta([0,1]^p)^n\Big]\le\log\Big(\frac{\beta p}{\beta p-\sigma}\Big)^{\frac{\beta p-\sigma}{\beta}}+\log\rho. \tag{2.25}$$

Combining (2.23) and (2.25) gives

$$\lim_{n\to\infty}\frac{1}{n}\log(n!)^{-\sigma/\beta}\mathbb{E}\Big[\zeta([0,1]^p)^n\Big]=\log\Big(\frac{\beta p}{\beta p-\sigma}\Big)^{\frac{\beta p-\sigma}{\beta}}+\log\rho. \tag{2.26}$$

Finally, Theorem 1.2 follows from Lemma 2.1. □

3 Lower bound for Theorem 1.2

In this section we prove

$$\liminf_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^p}\mathbb{E}\Big[\zeta([0,\tau_1]\times\cdots\times[0,\tau_p])^n\Big]\ge\log\rho. \tag{3.1}$$

Our starting point is (2.6). Let $q>1$ be the conjugate of p defined by $p^{-1}+q^{-1}=1$ and let f be a symmetric, continuous, and strictly positive function on $\mathbb{R}^d$ with $||f||_{q,\varphi_{d-\sigma}}=1$, where

$$||f||_{q,\varphi_{d-\sigma}}=\Big(\int_{\mathbb{R}^d}|f(\lambda)|^q\varphi_{d-\sigma}(\lambda)d\lambda\Big)^{1/q}. \tag{3.2}$$

We have

$$\begin{aligned}
&\Big(\int_{(\mathbb{R}^d)^n}\Big[\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n Q\Big(\sum_{j=1}^k\lambda_{\sigma(j)}\Big)\Big]^p\prod_{i=1}^n\varphi_{d-\sigma}(\lambda_i)d\lambda_i\Big)^{1/p} \\
&\ge\int_{(\mathbb{R}^d)^n}\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n Q\Big(\sum_{j=1}^k\lambda_{\sigma(j)}\Big)\prod_{i=1}^n f(\lambda_i)\varphi_{d-\sigma}(\lambda_i)d\lambda_i \\
&=n!\int_{(\mathbb{R}^d)^n}\prod_{k=1}^n Q\Big(\sum_{j=1}^k\lambda_j\Big)\prod_{i=1}^n f(\lambda_i)\varphi_{d-\sigma}(\lambda_i)d\lambda_i \\
&=n!\int_{(\mathbb{R}^d)^n}\prod_{k=1}^n f(\lambda_k-\lambda_{k-1})\varphi_{d-\sigma}(\lambda_k-\lambda_{k-1})Q(\lambda_k)d\lambda_1\cdots d\lambda_n
\end{aligned} \tag{3.3}$$

where we follow the convention that $\lambda_0 = 0$.

Define the linear operator T on $\mathcal{L}^2(\mathbb{R}^d)$ as

$$Tg(\lambda) = \sqrt{Q(\lambda)} \int_{\mathbb{R}^d} f(\gamma-\lambda)\varphi_{d-\sigma}(\gamma-\lambda)\sqrt{Q(\gamma)}g(\gamma)d\gamma, \quad g \in \mathcal{L}^2(\mathbb{R}^d). \tag{3.4}$$

To show that T is well defined and continuous on $\mathcal{L}^2(\mathbb{R}^d)$, we need only to prove that there is a constant $C > 0$ such that

$$\langle h, Tg\rangle \le C||g||_2||h||_2, \quad g, h \in \mathcal{L}^2(\mathbb{R}^d). \tag{3.5}$$

But

$$\begin{aligned}\langle h, Tg\rangle &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} f(\gamma-\lambda)\varphi_{d-\sigma}(\gamma-\lambda)\sqrt{Q(\lambda)}h(\lambda)\sqrt{Q(\gamma)}g(\gamma)d\lambda d\gamma \\ &= \int_{\mathbb{R}^d} f(\gamma)\varphi_{d-\sigma}(\gamma)d\gamma \int_{\mathbb{R}^d} \sqrt{Q(\lambda)}h(\lambda)\sqrt{Q(\lambda+\gamma)}g(\lambda+\gamma)d\lambda \\ &\le \left\{ \int_{\mathbb{R}^d} \varphi_{d-\sigma}(\gamma)\Big[\int_{\mathbb{R}^d} \sqrt{Q(\lambda)}h(\lambda)\sqrt{Q(\lambda+\gamma)}g(\lambda+\gamma)d\lambda\Big]^p d\gamma\right\}^{1/p}.\end{aligned} \tag{3.6}$$

Hence by (1.21) (with f, g and h being replaced by g, h and Q^{-1}, respectively), $\langle h, Tg\rangle \le ||Q||_{pd/\sigma}||g||_2||h||_2$.

In addition, one can see that $\langle h, Tg\rangle = \langle g, Th\rangle$ for any $g, h \in \mathcal{L}^2(\mathbb{R}^d)$. We now let g be a bounded and locally supported function on $\mathbb{R}^d$ with $||g||_2 = 1$. Then there is $\delta > 0$ such that $f, \varphi_{d-\sigma}, Q \ge \delta$ on the support of g. In addition, notice that $Q \le 1$. Thus,

$$\begin{aligned}&\int_{(\mathbb{R}^d)^n} \prod_{k=1}^n f(\lambda_k-\lambda_{k-1})\varphi_{d-\sigma}(\lambda_k-\lambda_{k-1})Q(\lambda_k)d\lambda_1\cdots d\lambda_n \\ &\ge \delta^3||g||_\infty^{-2}\int_{(\mathbb{R}^d)^n} d\lambda_1\cdots d\lambda_n g(\lambda_1) \\ &\times\Big(\prod_{k=2}^n \sqrt{Q(\lambda_{k-1})}f(\lambda_k-\lambda_{k-1})\varphi_{d-\sigma}(\lambda_k-\lambda_{k-1})\sqrt{Q(\lambda_k)}\Big)g(\lambda_n) \\ &= \delta^3||g||_\infty^{-2}\langle g, T^{n-1}g\rangle.\end{aligned} \tag{3.7}$$

Consider the spectral representation of the self-adjoint operator T:

$$\langle g, Tg \rangle = \int_{-\infty}^{\infty} \theta \mu_g(d\theta) \tag{3.8}$$

where $\mu_g(d\theta)$ is a probability measure on $\mathbb{R}$. Therefore

$$\langle g, T^{n-1}g \rangle = \int_{-\infty}^{\infty} \theta^{n-1} \mu_g(d\theta) \geq \Big(\int_{-\infty}^{\infty} \theta \mu_g(d\theta) \Big)^{n-1} = \langle g, Tg \rangle^{n-1} \tag{3.9}$$

where the second step follows from Jensen's inequality.

Hence,

$$\begin{aligned} &\liminf_{n\to\infty} \frac{1}{n} \log \frac{1}{n!} \Big(\int_{(\mathbb{R}^d)^n} \Big[\sum_{\sigma\in\Sigma_n} \prod_{k=1}^{n} Q\Big(\sum_{j=1}^{k} \lambda_{\sigma(j)} \Big) \Big]^p \prod_{i=1}^{n} \varphi_{d-\sigma}(\lambda_i) d\lambda_i \Big)^{1/p} \\ &\geq \log \langle g, Tg \rangle \\ &= \log \iint_{\mathbb{R}^d \times \mathbb{R}^d} f(\gamma - \lambda) \varphi_{d-\sigma}(\gamma - \lambda) \sqrt{Q(\lambda)} \sqrt{Q(\gamma)} g(\lambda) g(\gamma) d\lambda d\gamma \\ &= \log \int_{\mathbb{R}^d} f(\lambda) \varphi_{d-\sigma}(\lambda) \Big[\int_{\mathbb{R}^d} \sqrt{Q(\lambda+\gamma)} \sqrt{Q(\gamma)} g(\lambda+\gamma) g(\gamma) d\gamma \Big] d\lambda. \end{aligned} \tag{3.10}$$

Notice that the set of all bounded, locally supported g is dense in $\mathcal{L}^2(\mathbb{R}^d)$. Taking the supremum over g on the right hand sides gives

$$\begin{aligned} &\liminf_{n\to\infty} \frac{1}{n} \log \frac{1}{n!} \Big(\int_{(\mathbb{R}^d)^n} \Big[\sum_{\sigma\in\Sigma_n} \prod_{k=1}^{n} Q\Big(\sum_{j=1}^{k} \lambda_{\sigma(j)} \Big) \Big]^p \prod_{i=1}^{n} \varphi_{d-\sigma}(\lambda_i) d\lambda_i \Big)^{1/p} \\ &\geq \log \sup_{\|g\|_2=1} \int_{\mathbb{R}^d} f(\lambda) \varphi_{d-\sigma}(\lambda) \Big[\int_{\mathbb{R}^d} \sqrt{Q(\lambda+\gamma)} \sqrt{Q(\gamma)} g(\lambda+\gamma) g(\gamma) d\gamma \Big] d\lambda. \end{aligned} \tag{3.11}$$

Notice that for any g, the function

$$H(\lambda) = \int_{\mathbb{R}^d} \sqrt{Q(\lambda+\gamma)} \sqrt{Q(\gamma)} g(\lambda+\gamma) g(\gamma) d\gamma \tag{3.12}$$

is symmetric: $H(-\lambda) = H(\lambda)$. Hence, taking the supremum over all symmetric, continuous, and strictly positive functions f with $||f||_{q,\varphi_{d-\sigma}} = 1$ (Recall that the norm $||\cdot||_{q,\varphi_{d-\sigma}}$ is defined in (3.2)) on the right gives

$$\liminf_{n\to\infty}\frac{1}{n}\log\frac{1}{n!}\Big(\int_{(\mathbb{R}^d)^n}\Big[\sum_{\sigma\in\Sigma_n}\prod_{k=1}^{n}Q\Big(\sum_{j=1}^{k}\lambda_{\sigma(j)}\Big)\Big]^p\prod_{i=1}^{n}\varphi_{d-\sigma}(\lambda_i)d\lambda_i\Big)^{1/p} \tag{3.13}$$
$$\geq\frac{1}{p}\log\sup_{\|g\|_2=1}\int_{\mathbb{R}^d}\varphi_{d-\sigma}(\lambda)\Big[\int_{\mathbb{R}^d}\sqrt{Q(\lambda+\gamma)}\sqrt{Q(\gamma)}g(\lambda+\gamma)g(\gamma)d\gamma\Big]^p d\lambda$$
$$=\frac{1}{p}\log\rho.$$

From the relation (2.6), we have proved (3.1). □

4 Proof of Lemma 2.2

Before proving the upper bound for Theorem 2.1 we provide the proof of Lemma 2.2, since we will need several easy generalizations of this proof. In the course of our proof we will use certain Sobolev-type inequalities which are proven in the Appendix.

We first look at

$$\mathbb{E}\left(\prod_{j=1}^{n}\int_0^{\tau_1}|X_1(s_j)+a_j|^{-\sigma}ds_j\right) \tag{4.1}$$
$$=\sum_{\pi\in\Sigma_n}\mathbb{E}\left(\int_{0\leq s_{\pi(1)}\leq\cdots\leq s_{\pi(n)}\leq\tau_1}\prod_{j=1}^{n}|X_1(s_j)+a_j|^{-\sigma}ds_1\cdots ds_n\right)$$
$$=\sum_{\pi\in\Sigma_n}\mathbb{E}\Bigg(\int_{0\leq s_{\pi(1)}\leq\cdots\leq s_{\pi(n)}\leq\tau_1}$$
$$\left(\int_{(\mathbb{R}^d)^n}\prod_{j=1}^{n}|x_{\pi(j)}+a_{\pi(j)}|^{-\sigma}p_{s_{\pi(j)}-s_{\pi(j-1)}}(x_{\pi(j)}-x_{\pi(j-1)})\,dx_j\right)ds_1\cdots ds_n\Bigg)$$
$$=\sum_{\pi\in\Sigma_n}\int_{(\mathbb{R}^d)^n}\prod_{j=1}^{n}|x_{\pi(j)}+a_{\pi(j)}|^{-\sigma}\prod_{j=1}^{n}u^1(x_{\pi(j)}-x_{\pi(j-1)})\,dx_j.$$

Here we recall that the notation $u^1(x)$ comes from (1.30).

Similarly, proceeding inductively we obtain

$$\mathbb{E}\left(\prod_{j=1}^{n}\int_0^{\tau_1}\cdots\int_0^{\tau_p}|X_1(s_{1,j})+\cdots+X_p(s_{p,j})-z|^{-\sigma}\prod_{l=1}^{p}ds_{l,j}\right)$$
$$=\sum_{\pi_1,\ldots,\pi_p\in\Sigma_n}\int_{(\mathbb{R}^d)^{np}}\prod_{j=1}^{n}|x_{1,\pi_1(j)}+\cdots+x_{p,\pi_p(j)}-z|^{-\sigma}$$
$$\times\prod_{l=1}^{p}\prod_{j=1}^{n}u^1(x_{l,\pi_l(j)}-x_{l,\pi_l(j-1)})\,dx_{l,j}. \tag{4.2}$$

For $f\in\mathcal{S}(\mathbb{R}^d)$ let us consider

$$\sum_{\pi_1,\ldots,\pi_p\in\Sigma_n}\int_{(\mathbb{R}^d)^{np}}\prod_{j=1}^{n}|x_{1,\pi_1(j)}+\cdots+x_{p,\pi_p(j)}-z|^{-\sigma}$$
$$\times\prod_{l=1}^{p}\prod_{j=1}^{n}f(x_{l,\pi_l(j)}-x_{l,\pi_l(j-1)})\,dx_{l,j}. \tag{4.3}$$

By (1.37)

$$\int_{\mathbb{R}^d}|x|^{-\sigma}f(x)\,dx=\int_{\mathbb{R}^d}\varphi_{d-\sigma}(\lambda)\widehat{f}(\lambda)\,d\lambda \tag{4.4}$$

and hence

$$\int_{\mathbb{R}^d}|x+a|^{-\sigma}f(x)\,dx=\int_{\mathbb{R}^d}e^{i\lambda\cdot a}\varphi_{d-\sigma}(\lambda)\widehat{f}(\lambda)\,d\lambda$$
$$=\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}e^{i\lambda\cdot(x+a)}f(x)\,dx\right)\varphi_{d-\sigma}(\lambda)\,d\lambda. \tag{4.5}$$

Therefore

$$
\begin{aligned}
(4.6)\quad &\sum_{\pi_1,\dots,\pi_p\in\Sigma_n}\int_{(\mathbb{R}^d)^{np}}\prod_{j=1}^n |x_{1,\pi_1(j)}+\cdots+x_{p,\pi_p(j)}-z|^{-\sigma}\\
&\qquad\qquad\times\prod_{l=1}^p\prod_{j=1}^n f(x_{l,\pi_l(j)}-x_{l,\pi_l(j-1)})\,dx_{l,j}.\\
&=\sum_{\pi_1,\dots,\pi_p\in\Sigma_n}\int_{(\mathbb{R}^d)^n} e^{-i\sum_{j=1}^n\lambda_j\cdot z}\left(\int e^{i\sum_{j=1}^n\lambda_j\cdot(x_{1,\pi_1(j)}+\cdots+x_{p,\pi_p(j)})}\right.\\
&\qquad\qquad\left.\times\prod_{l=1}^p\prod_{j=1}^n f(x_{l,\pi_l(j)}-x_{l,\pi_l(j-1)})\,dx_{l,j}\right)\prod_{j=1}^n\varphi_{d-\sigma}(\lambda_j)\,d\lambda_j.\\
&=\sum_{\pi_1,\dots,\pi_p\in\Sigma_n}\int_{(\mathbb{R}^d)^n}\prod_{l=1}^p\left(\int_{(\mathbb{R}^d)^n} e^{i\sum_{j=1}^n\lambda_j\cdot x_{l,\pi_l(j)}}\prod_{j=1}^n f(x_{l,\pi_l(j)}-x_{l,\pi_l(j-1)})\,dx_{l,j}\right)\\
&\qquad\qquad\times e^{-i\sum_{j=1}^n\lambda_j\cdot z}\prod_{j=1}^n\varphi_{d-\sigma}(\lambda_j)\,d\lambda_j.
\end{aligned}
$$

Note that

$$
\begin{aligned}
(4.7)\quad &\int_{(\mathbb{R}^d)^n} e^{i\sum_{j=1}^n\lambda_j\cdot x_{\pi(j)}}\prod_{j=1}^n f(x_{\pi(j)}-x_{\pi(j-1)})\,dx_j\\
&=\int_{(\mathbb{R}^d)^n} e^{i\sum_{j=1}^n\lambda_{\pi^{-1}(j)}\cdot x_j}\prod_{j=1}^n f(x_j-x_{j-1})\,dx_j\\
&=\int_{(\mathbb{R}^d)^n} e^{i\sum_{j=1}^n(\sum_{k=j}^n\lambda_{\pi^{-1}(k)})\cdot x_j}\prod_{j=1}^n f(x_j)\,dx_j\\
&=\prod_{j=1}^n\widehat{f}\left(\sum_{k=j}^n\lambda_{\pi^{-1}(k)}\right)=\prod_{j=1}^n\widehat{f}\left(\sum_{k=1}^j\lambda_{\pi'(k)}\right)
\end{aligned}
$$

with π' defined so that $\pi'(j) = \pi^{-1}(n-j)$, $\forall j$. Hence we obtain

$$
\begin{aligned}
(4.8)\quad & \sum_{\pi_1,\ldots,\pi_p\in\Sigma_n} \int_{(\mathbb{R}^d)^{np}} \prod_{j=1}^n |x_{1,\pi_1(j)} + \cdots + x_{p,\pi_p(j)} - z|^{-\sigma} \\
& \qquad\qquad \times \prod_{l=1}^p \prod_{j=1}^n f(x_{l,\pi_l(j)} - x_{l,\pi_l(j-1)})\, dx_{l,j} \\
&= \sum_{\pi_1,\ldots,\pi_p\in\Sigma_n} \int_{(\mathbb{R}^d)^n} e^{-i\sum_{j=1}^n \lambda_j\cdot z} \prod_{l=1}^p \left(\prod_{j=1}^n \widehat{f}\left(\sum_{k=1}^j \lambda_{\pi_l(k)} \right)\right) \prod_{j=1}^n \varphi_{d-\sigma}(\lambda_j)\, d\lambda_j \\
&= \int_{(\mathbb{R}^d)^n} e^{-i\sum_{j=1}^n \lambda_j\cdot z} \left[\sum_{\pi\in\Sigma_n} \prod_{j=1}^n \widehat{f}\left(\sum_{k=1}^j \lambda_{\pi(k)} \right)\right]^p \prod_{j=1}^n \varphi_{d-\sigma}(\lambda_j)\, d\lambda_j.
\end{aligned}
$$

Assuming that f, $\widehat{f} \geq 0$ we see as in (2.16) that

$$
\begin{aligned}
(4.9)\qquad & \int_{(\mathbb{R}^d)^n} \left[\sum_{\pi\in\Sigma_n} \prod_{j=1}^n \widehat{f}\left(\sum_{k=1}^j \lambda_{\pi(k)} \right)\right]^p \prod_{j=1}^n \varphi_{d-\sigma}(\lambda_j)\, d\lambda_j \\
& \leq (n!)^p \left(\int_{\mathbb{R}^d} \varphi_{d-\sigma}(\lambda) \left(\widehat{f}(\lambda)\right)^p d\lambda \right)^n
\end{aligned}
$$

and by (4.8) with $n = 1$

$$
(4.10)\qquad \int_{\mathbb{R}^d} \varphi_{d-\sigma}(\lambda) \left(\widehat{f}(\lambda)\right)^p d\lambda = \int_{\mathbb{R}^d} \frac{1}{|x_1 + \cdots + x_p|^\sigma} \prod_{j=1}^p f(x_j)\, dx_j.
$$

By (10.2) with σ replaced by $d - \sigma$

$$
(4.11)\qquad \int_{(\mathbb{R}^d)^p} \frac{1}{|x_1 + \cdots + x_p|^\sigma} \prod_{j=1}^p f(x_j)\, dx_j \leq C^p \|f\|^p_{pd/(pd-\sigma)}.
$$

Now, $u^1(x)$ is integrable, monotone decreasing in $|x|$ and asymptotic at $x = 0$ to $u^0(x) = C|x|^{-\max(0,(d-\beta))}$. Hence

$$
(4.12)\qquad \|u^1\|_{pd/(pd-\sigma)} < \infty
$$

if $(d-\beta)pd/(pd-\sigma) < d$ which follows from (1.4). Choose some $g \in \mathcal{S}(\mathbb{R}^d)$ with $\int g(x)\, dx = 1$ such that both g and $\widehat{g}$ are positive, spherically symmetric and decreasing. (For example, we can take g to be the standard normal density). Set $g_\epsilon(x) = \epsilon^{-d} g(x/\epsilon)$. For any sequence $\epsilon_r \to 0$ let $f_r = g_{\epsilon_r} * (u^1 \widehat{g_{\epsilon_r}}) \in \mathcal{S}(R^d)$. We see that

$$\lim_{r\to\infty} \|u^1 - f_r\|_{pd/(pd-\sigma)} = 0 \tag{4.13}$$

and $\widehat{f_r} = \widehat{g_{\epsilon_r}}(\widehat{u^1} * g_{\epsilon_r})$ converges pointwise to $\widehat{u^1}$. Using (2.17) we can see that $\widehat{f_r}$ is positive, spherically symmetric and decreasing. In view of (4.2) and (4.8), to prove (2.6) it suffices to show that

$$\begin{aligned} \lim_{r\to\infty} \sum_{\pi_1,\ldots,\pi_p \in \Sigma_n} \int_{(\mathbb{R}^d)^{np}} \prod_{j=1}^n |x_{1,\pi_1(j)} + \cdots + x_{p,\pi_p(j)} - z|^{-\sigma} \\ \times \prod_{l=1}^p \prod_{j=1}^n f_r(x_{l,\pi_l(j)} - x_{l,\pi_l(j-1)})\, dx_{l,j} \\ = \sum_{\pi_1,\ldots,\pi_p \in \Sigma_n} \int_{(\mathbb{R}^d)^{np}} \prod_{j=1}^n |x_{1,\pi_1(j)} + \cdots + x_{p,\pi_p(j)} - z|^{-\sigma} \\ \times \prod_{l=1}^p \prod_{j=1}^n u^1(x_{l,\pi_l(j)} - x_{l,\pi_l(j-1)})\, dx_{l,j} \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} \lim_{r\to\infty} \int_{(\mathbb{R}^d)^n} e^{-i\sum_{j=1}^n \lambda_j\cdot z} \left[\sum_{\pi\in\Sigma_n} \prod_{j=1}^n \widehat{f_r}\left(\sum_{k=1}^j \lambda_{\pi(k)} \right) \right]^p \prod_{j=1}^n \varphi_{d-\sigma}(\lambda_j)\, d\lambda_j \\ = \int_{(\mathbb{R}^d)^n} e^{-i\sum_{j=1}^n \lambda_j\cdot z} \left[\sum_{\pi\in\Sigma_n} \prod_{j=1}^n \widehat{u^1}\left(\sum_{k=1}^j \lambda_{\pi(k)} \right) \right]^p \prod_{j=1}^n \varphi_{d-\sigma}(\lambda_j)\, d\lambda_j. \end{aligned} \tag{4.15}$$

For fixed $\pi_1, \ldots, \pi_p \in \Sigma_n$, the difference between integral on the the right hand side of (4.14) and the left hand side of (4.14) for fixed r is

$$\int_{(\mathbb{R}^d)^{np}} \prod_{j=1}^n |x_{1,\pi_1(j)} + \cdots + x_{p,\pi_p(j)} - z|^{-\sigma} F_r \prod_{l=1}^p \prod_{j=1}^n dx_{l,j} \tag{4.16}$$

with

$$F_r = \prod_{l=1}^p \prod_{j=1}^n u^1(x_{l,\pi_l(j)} - x_{l,\pi_l(j-1)}) - \prod_{l=1}^p \prod_{j=1}^n f_r(x_{l,\pi_l(j)} - x_{l,\pi_l(j-1)}). \tag{4.17}$$

Writing $A_{(l-1)n+j} = u^1(x_{l,\pi_l(j)} - x_{l,\pi_l(j-1)})$, $B_{(l-1)n+j} = f_r(x_{l,\pi_l(j)} - x_{l,\pi_l(j-1)})$, we can write

$$F_r = \prod_{s=1}^{np} A_s - \prod_{s=1}^{np} B_{r,s} = \sum_{t=1}^{np} \prod_{s=1}^{t-1} A_s \left(A_t - B_{r,t}\right) \prod_{s=t+1}^{np} B_{r,s}. \tag{4.18}$$

It suffices to show that

$$\Bigg| \int_{(\mathbb{R}^d)^{np}} \prod_{j=1}^{n} |x_{1,\pi_1(j)} + \cdots + x_{p,\pi_p(j)} - z|^{-\sigma} \times \left\{ \prod_{s=1}^{t-1} A_s \left(A_t - B_{r,t}\right) \prod_{s=t+1}^{np} B_{r,s} \right\} \prod_{l=1}^{p} \prod_{j=1}^{n} dx_{l,j} \Bigg| \tag{4.19}$$

goes to 0 as $r \to \infty$. It is easy to see that the product in brackets can be written in the form needed for (10.3). More precisely,

$$\prod_{s=1}^{t-1} A_s \left(A_t - B_{r,t}\right) \prod_{s=t+1}^{np} B_{r,s} = \prod_{l=1}^{p} H_l \tag{4.20}$$

with

$$H_l = \prod_{j=1}^{n} h_{l,j}(x_{l,\pi_l(j)} - x_{l,\pi_l(j-1)}) \tag{4.21}$$

where

$$h_{l,j} = \begin{cases} u^1 & \text{if } (l-1)n + j < t \\ u^1 - f_r & \text{if } (l-1)n + j = t \\ f_r & \text{if } (l-1)n + j > t. \end{cases}$$

By (10.3) with σ replaced by $d - \sigma$, we see that (4.19) is bounded by

$$\begin{aligned} & C \prod_{l=1}^{p} \| \prod_{j=1}^{n} h_{l,j}(x_{l,\pi_l(j)} - x_{l,\pi_l(j-1)}) \|_{pd/(pd-\sigma)} \\ &= C \prod_{l=1}^{p} \| \prod_{j=1}^{n} h_{l,j}(x_{l,\pi_l(j)}) \|_{pd/(pd-\sigma)} \\ &= C \prod_{l=1}^{p} \prod_{j=1}^{n} \| h_{l,j} \|_{pd/(pd-\sigma)}. \end{aligned} \tag{4.22}$$

Using (4.12) and (4.13) it is easy to see that this goes to 0 as $r \to \infty$, completing the proof of (4.14).

Let $\|f\|_{p,\varphi_{d-\sigma},n}$ denote the L^p norm on $\mathbb{R}^{dn}$ with respect to the measure $\prod_{j=1}^n \varphi_{d-\sigma}(\lambda_j)\, d\lambda_j$. For any $\pi \in \Sigma_n$, any $1 \le j \le n$ and any function h on $\mathbb{R}^d$ set

$$h_\pi(\lambda_1, \cdots, \lambda_n) = \prod_{j=1}^n h\left(\sum_{k=1}^j \lambda_{\pi(k)} \right)$$

so that

$$(4.23) \int_{(\mathbb{R}^d)^n} \left[\sum_{\pi\in\Sigma_n} \prod_{j=1}^n h\left(\sum_{k=1}^j \lambda_{\pi(k)} \right) \right]^p \prod_{j=1}^n \varphi_{d-\sigma}(\lambda_j)\, d\lambda_j = \left\| \sum_{\pi\in\Sigma_n} h_\pi \right\|_{p,\varphi_{d-\sigma},n}^p .$$

Then the absolute value of the difference between the left-hand side of (4.15) for fixed r and the right hand side of (4.15) is bounded by

$$\left\| \sum_{\pi\in\Sigma_n} \widehat{u^1}_\pi - \widehat{(f_r)}_\pi \right\|_{p,\varphi_{d-\sigma},n}^p$$

and

$$\begin{aligned} &\left\| \sum_{\pi\in\Sigma_n} \widehat{u^1}_\pi - \widehat{(f_r)}_\pi \right\|_{p,\varphi_{d-\sigma},n} \\ &\le n! \|\widehat{u^1}_{id} - \widehat{(f_r)}_{id}\|_{p,\varphi_{d-\sigma},n} \\ &\le n! \sum_{m=1}^n \left\| \left\{ \prod_{j=1}^{m-1} \mathcal{U}_j \right\} |\mathcal{U}_m - \mathcal{F}_m)| \left\{ \prod_{j=m+1}^{n} \mathcal{F}_j \right\} \right\|_{p,\varphi_{d-\sigma},n} \end{aligned}$$

where for each $1 \le j \le n$,

$$\mathcal{U}_j(\lambda_1, \cdots, \lambda_j) = \widehat{u^1}\left(\sum_{k=1}^j \lambda_k \right), \qquad \lambda_1, \cdots, \lambda_j \in \mathbb{R}^d$$

$$\mathcal{F}_j(\lambda_1, \cdots, \lambda_j) = \widehat{u^1}\left(\sum_{k=1}^j \lambda_k \right), \qquad \lambda_1, \cdots, \lambda_j \in \mathbb{R}^d.$$

By (2.17)

$$\begin{aligned}
&\Big\| \Big\{ \prod_{j=1}^{m-1} \mathcal{U}_j \Big\} |\mathcal{U}_m - \mathcal{F}_m)| \Big\{ \prod_{j=m+1}^{n} \mathcal{F}_j \Big\} \Big\|_{p,\varphi_{d-\sigma},n} \\
&= \int_{(\mathbb{R}^d)^n} \Big\{ \prod_{j=1}^{m-1} \varphi_{d-\sigma}(\lambda_{j-1}-\lambda_j) \left(\widehat{u^1}(\lambda_j)\right)^p \Big\} \varphi_{d-\sigma}(\lambda_{m-1}-\lambda_m) \Big| \left(\widehat{u^1} - \widehat{f}_r\right)(\lambda_m) \Big|^p \\
&\qquad \times \Big\{ \prod_{j=m+1}^{n} \varphi_{d-\sigma}(\lambda_{j-1}-\lambda_j) \left(\widehat{f}_r(\lambda_j)\right)^p \Big\} \, d\lambda_1 \cdots d\lambda_n \\
&\le \left(\int_{\mathbb{R}^d} \varphi_{d-\sigma}(\lambda) \left(\widehat{f}_r(\lambda)\right)^p d\lambda \right)^{n-m} \int \Big\{ \prod_{j=1}^{m-1} \varphi_{d-\sigma}(\lambda_{j-1}-\lambda_j) \left(\widehat{u^1}(\lambda_j)\right)^p \Big\} \\
&\qquad \varphi_{d-\sigma}(\lambda_{m-1}-\lambda_m) \Big| \left(\widehat{u^1} - \widehat{f}_r\right)(\lambda_m) \Big|^p d\lambda_1 \cdots d\lambda_m. \qquad (4.24)
\end{aligned}$$

As in (4.10)–(4.11),

$$\int_{\mathbb{R}^d} \varphi_{d-\sigma}(\lambda) \left(\widehat{f}_r(\lambda)\right)^p d\lambda \le C \|f_r\|^p_{pd/(pd-\sigma)} \qquad (4.25)$$

so it remains to show that

$$\begin{aligned}
\lim_{r\to\infty} \int_{(\mathbb{R}^d)^m} &\Big\{ \prod_{j=1}^{m-1} \varphi_{d-\sigma}(\lambda_{j-1}-\lambda_j) \left(\widehat{u^1}(\lambda_j)\right)^p \Big\} \\
&\times \varphi_{d-\sigma}(\lambda_{m-1}-\lambda_m) \Big| \left(\widehat{u^1} - \widehat{f}_r\right)(\lambda_m) \Big|^p d\lambda_1 \cdots d\lambda_m = 0. \qquad (4.26)
\end{aligned}$$

We use the uniform integrability of

$$G_r =: \prod_{j=1}^{m-1} \left(\widehat{u^1}(\lambda_j)\right)^p \Big| \left(\widehat{u^1} - \widehat{f}_r\right)(\lambda_m) \Big|^p$$

with respect to the measure $d\mu = \prod_{j=1}^m \varphi_{d-\sigma}(\lambda_{j-1}-\lambda_j) d\lambda_j$. To see that G_r

is uniformly integrable it suffices to show that for some $\epsilon > 0$

$$(4.27)\int G_r^{1+\epsilon}\,d\mu = \int_{(\mathbb{R}^d)^m} \left\{ \prod_{j=1}^{m-1} \varphi_{d-\sigma}(\lambda_{j-1}-\lambda_j)\left(\widehat{u^1}(\lambda_j)\right)^{p(1+\epsilon)} \right\}$$

$$\times\varphi_{d-\sigma}(\lambda_{m-1}-\lambda_m)\left|\left(\widehat{u^1}-\widehat{f}_r\right)(\lambda_m)\right|^{p(1+\epsilon)} d\lambda_1\cdots d\lambda_m$$

$$\leq \int_{(\mathbb{R}^d)^m} \left\{ \prod_{j=1}^{m-1} \varphi_{d-\sigma}(\lambda_{j-1}-\lambda_j)\left(\widehat{u^1}(\lambda_j)\right)^{p(1+\epsilon)} \right\}$$

$$\times\varphi_{d-\sigma}(\lambda_{m-1}-\lambda_m)\left\{\left(\widehat{u^1}(\lambda_m)\right)^{p(1+\epsilon)} + \left(\widehat{f}_r(\lambda_m)\right)^{p(1+\epsilon)}\right\} d\lambda_1\cdots d\lambda_m$$

is bounded uniformly in r and this follows as before, using (2.17), (4.25) and (4.10)–(4.13). Since $\lim_{r\to\infty} G_r = 0$ we see that (4.26) holds and this establishes (4.15).

Let

$$(4.28)\qquad u_{n,t}(y_1,\ldots,y_n) = \int_{0\leq s_1\leq\cdots\leq s_n\leq t} \prod_{j=1}^n p_{s_j-s_{j-1}}(y_j-y_{j-1})ds_j.$$

To prove (2.7) we first note as in (4.1)

$$(4.29)\qquad \mathbb{E}\left(\prod_{j=1}^n \int_0^{t_1} |X_1(s_j)+a_j|^{-\sigma} ds_j\right)$$

$$= \sum_{\pi\in\Sigma_n} \mathbb{E}\left(\int_{0\leq s_{\pi(1)}\leq\cdots\leq s_{\pi(n)}\leq t_1} \prod_{j=1}^n |X_1(s_j)+a_j|^{-\sigma} ds_1\cdots ds_n\right)$$

$$= \sum_{\pi\in\Sigma_n} \int_{(\mathbb{R}^d)^n} \prod_{j=1}^n |x_{\pi(j)}+a_{\pi(j)}|^{-\sigma}\;\; u_{n,t_1}(x_{\pi(1)},\ldots,x_{\pi(n)})\,dx_j.$$

Similarly, proceeding inductively we obtain

$$(4.30)\qquad \mathbb{E}\left(\prod_{j=1}^n \int_0^{t_1}\cdots\int_0^{t_p} |X_1(s_{1,j})+\cdots+X_p(s_{p,j})-z|^{-\sigma} \prod_{l=1}^p ds_{l,j}\right)$$

$$= \sum_{\pi_1,\ldots,\pi_p\in\Sigma_n} \int_{(\mathbb{R}^d)^{np}} \prod_{j=1}^n |x_{1,\pi_1(j)}+\cdots+x_{p,\pi_p(j)}-z|^{-\sigma}$$

$$\times\prod_{l=1}^p u_{n,t_l}(x_{\pi_l(1)},\ldots,x_{\pi_l(n)}) \prod_{l=1}^p\prod_{j=1}^n dx_{l,j}.$$

Then as before we can show that

$$(4.31)\quad \sum_{\pi_1,\dots,\pi_p\in\Sigma_n}\int_{(\mathbb{R}^d)^{np}}\prod_{j=1}^n |x_{1,\pi_1(j)}+\cdots+x_{p,\pi_p(j)}-z|^{-\sigma}$$
$$\times\prod_{l=1}^p u_{n,t_l}(x_{\pi_l(1)},\dots,x_{\pi_l(n)})\prod_{l=1}^p\prod_{j=1}^n dx_{l,j}$$
$$=\int_{(\mathbb{R}^d)^n}\prod_{l=1}^p e^{-i\sum_{j=1}^n\lambda_j\cdot z}\left[\sum_{\pi\in\Sigma_n}F_{n,t_l}\left(\lambda_{\pi(1)},\dots,\sum_{k=1}^n\lambda_{\pi(k)}\right)\right]\prod_{j=1}^n\varphi_{d-\sigma}(\lambda_j)\,d\lambda_j$$

where

$$(4.32)\quad \begin{aligned}&F_{n,t}\left(\lambda_1,\dots,\lambda_n\right)\\ &=\int e^{i\sum_{j=1}^n\lambda_j\cdot y_j}\int_{0\le s_1\le\cdots\le s_n\le t}\prod_{j=1}^n p_{s_j-s_{j-1}}(y_j)\,ds_j\\ &=\int_{0\le s_1\le\cdots\le s_n\le t}\prod_{j=1}^n e^{-(s_j-s_{j-1})\psi(\lambda_j)}\,ds_j\end{aligned}$$

which is non-negative. It then follows from the generalized Hölder's inequality that

$$(4.33)\quad \mathbb{E}\Big[\zeta([0,t_1]\times\cdots\times[0,t_p])^m\Big]\le\prod_{l=1}^p\Big(\mathbb{E}\Big[\zeta([0,t_l]^p)^m\Big]\Big)^{1/p}$$

and (2.7) then follows from the scaling relation (1.8). □

For future reference we note that (4.31) and the fact that $F_{n,t}$ is non-negative shows that

$$(4.34)\quad \sup_z\mathbb{E}\Big[\zeta^z([0,t_1]\times\cdots\times[0,t_p])^m\Big]=\mathbb{E}\Big[\zeta([0,t_1]\times\cdots\times[0,t_p])^m\Big].$$

5 Upper bound for Theorem 1.2

In this section we prove

$$(5.1)\quad \limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^p}\mathbb{E}\Big[\zeta([0,\tau_1]\times\cdots\times[0,\tau_p])^n\Big]\le\log\rho.$$

Define the probability density h on $\mathbb{R}^d$ as

$$h(x) = C^{-1} \prod_{j=1}^{d} \Big(\frac{2\sin x_j}{x_j}\Big)^2, \quad x = (x_1, \cdots, x_d) \in \mathbb{R}^d \tag{5.2}$$

where $C > 0$ is the normalizing constant:

$$C = \int_{\mathbb{R}^d} \prod_{j=1}^{d} \Big(\frac{2\sin x_k}{x_k}\Big)^2 dx_1 \cdots dx_d.$$

Clearly, h is symmetric. One can verify that the Fourier transform $\widehat{h}$ is

$$\widehat{h}(\lambda) = \int_{\mathbb{R}^d} h(x) e^{i\lambda\cdot x} dx = C^{-1}(2\pi)^d \Big(1_{[-1,1]^d} * 1_{[-1,1]^d}\Big)(\lambda).$$

In particular, $\widehat{h}$ is non-negative, continuous, with compact support in the set $[-2,2]^d$, and

$$\widehat{h}(\lambda) \le \widehat{h}(0) = 1. \tag{5.3}$$

For each $\epsilon > 0$, write

$$h_\epsilon(x) = \epsilon^{-d} h(\epsilon^{-1}x), \quad x \in \mathbb{R}^d.$$

For some constant $k_{d,\sigma}$ we have

$$\int_{\mathbb{R}^d} \frac{k_{d,\sigma}}{|s-\lambda|^{d-(\sigma/2)}} \frac{k_{d,\sigma}}{|s|^{d-(\sigma/2)}}\, ds = \frac{C_{d,\sigma}}{|\lambda|^{d-\sigma}} = \varphi_{d-\sigma}(\lambda). \tag{5.4}$$

Let

$$\wp_{\beta,\epsilon}(\lambda) = \frac{k_{d,\sigma}\widehat{h}(\epsilon\lambda)}{\beta + |\lambda|^{d-(\sigma/2)}} \tag{5.5}$$

and note that by (5.3) and (5.4)

$$\wp_{\beta,\epsilon} * \wp_{\beta,\epsilon}(\lambda) \le \wp_{\beta,0} * \wp_{\beta,0}(\lambda) = \varphi_{d-\sigma}(\lambda). \tag{5.6}$$

Let

$$\theta_{\beta,\epsilon}(x) = \int_{\mathbb{R}^d} e^{ix\cdot\lambda} \frac{k_{d,\sigma}\widehat{h}(\epsilon\lambda)}{\beta + |\lambda|^{d-\sigma/2}}\, d\lambda = \int_{\mathbb{R}^d} e^{ix\cdot\lambda} \wp_{\beta,\epsilon}(\lambda)\, d\lambda. \tag{5.7}$$

Then

$$\theta^2_{\beta,\epsilon}(x) = \int_{\mathbb{R}^d} e^{ix\cdot\lambda}\wp_{\beta,\epsilon} * \wp_{\beta,\epsilon}(\lambda)\, d\lambda. \tag{5.8}$$

Define

$$\zeta_{\beta,\epsilon}\big([0,t_1]\times\cdots\times[0,t_p]\big) = \int_0^{t_1}\cdots\int_0^{t_p}\theta^2_{\beta,\epsilon}(X_1(s_1)+\cdots+X_p(s_p))ds_1\cdots ds_p. \tag{5.9}$$

(5.1) will follow from the next two Lemmas.

Lemma 5.1

$$\limsup_{\beta,\epsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^p}\mathbb{E}\Big[(\zeta-\zeta_{\beta,\epsilon})\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\Big]^n = -\infty. \tag{5.10}$$

Lemma 5.2

$$\limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^p}\mathbb{E}\Big[\zeta_{\beta,\epsilon}([0,\tau_1]\times\cdots\times[0,\tau_p])^n\Big]\le\log\rho. \tag{5.11}$$

Proof of Lemma 5.1.

By (5.8) and (5.9)

$$\begin{aligned}&\zeta_{\beta,\epsilon}\big([0,t_1]\times\cdots\times[0,t_p]\big)\\ &=\int_{\mathbb{R}^d}\bigg(\int_0^{t_1}\cdots\int_0^{t_p}\exp\Big\{i\lambda\cdot\big(X_1(s_1)+\cdots+X_p(s_p)\big)\Big\}ds_1\cdots ds_p\bigg)\\ &\qquad\qquad\wp_{\beta,\epsilon}*\wp_{\beta,\epsilon}(\lambda)\,d\lambda.\end{aligned} \tag{5.12}$$

Following the same procedure used for (2.6),

$$\begin{aligned}&\mathbb{E}\Big[(\zeta-\zeta_{\beta,\epsilon})\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^n\Big]\\ &=\int_{(\mathbb{R}^d)^n}\Big[\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n Q\Big(\sum_{j=1}^k\lambda_{\sigma(j)}\Big)\Big]^p\prod_{k=1}^n\big[\varphi_{d-\sigma}(\lambda_k)-\wp_{\beta,\epsilon}*\wp_{\beta,\epsilon}(\lambda_k)\big]\,d\lambda_k\end{aligned} \tag{5.13}$$

where $Q(\lambda)=\big[1+\psi(\lambda)\big]^{-1}$.

Note that

$$(5.14)\quad \begin{aligned} 0 &\le \varphi_{d-\sigma}(\lambda) - \wp_{\beta,\epsilon} * \wp_{\beta,\epsilon}(\lambda) = \\ &= \left(\varphi_{d-\sigma}(\lambda) - \wp_{\beta,0} * \wp_{\beta,0}(\lambda)\right) + \left(\wp_{\beta,0} * \wp_{\beta,0}(\lambda) - \wp_{\beta,\epsilon} * \wp_{\beta,\epsilon}(\lambda)\right). \end{aligned}$$

By (5.4) we have

$$(5.15)\quad \begin{aligned} 0 &\le \varphi_{d-\sigma}(\lambda) - \wp_{\beta,0} * \wp_{\beta,0}(\lambda) \\ &= k_{d,\sigma}^2 \Big(\int_{\mathbb{R}^d} \frac{1}{|s-\lambda|^{d-(\sigma/2)}} \, \frac{1}{|s|^{d-(\sigma/2)}} \, ds \\ &\qquad - \int_{\mathbb{R}^d} \frac{1}{\beta + |s-\lambda|^{d-(\sigma/2)}} \, \frac{1}{\beta + |s|^{d-(\sigma/2)}} \, ds \Big) \\ &\le C \frac{\beta^\delta}{|\lambda|^{d-(\sigma/2)+\delta}}. \end{aligned}$$

We also have

$$(5.16)\quad \begin{aligned} &0 \le \wp_{\beta,0} * \wp_{\beta,0}(\lambda) - \wp_{\beta,\epsilon} * \wp_{\beta,\epsilon}(\lambda) \\ &= k_{d,\sigma}^2 \left(\int_{\mathbb{R}^d} \frac{1}{\beta + |s-\lambda|^{d-(\sigma/2)}} \, \frac{1}{\beta + |s|^{d-(\sigma/2)}} \, ds \right. \\ &\qquad \left. - \int_{\mathbb{R}^d} \frac{\widehat{h}(\epsilon(s-\lambda))}{\beta + |s-\lambda|^{d-(\sigma/2)}} \, \frac{\widehat{h}(\epsilon s)}{\beta + |s|^{d-(\sigma/2)}} \, ds \right) \\ &\le k_{d,\sigma}^2 \left(\int \frac{1}{|s-\lambda|^{d-(\sigma/2)}} \, \frac{1 - \widehat{h}(\epsilon s)}{|s|^{d-(\sigma/2)}} \, ds \right. \\ &\qquad \left. + \int_{\mathbb{R}^d} \frac{1 - \widehat{h}(\epsilon(s-\lambda))}{|s-\lambda|^{d-(\sigma/2)}} \, \frac{1}{|s|^{d-(\sigma/2)}} \, ds \right). \end{aligned}$$

Fix $\gamma > 0$ and choose $\tau > 0$ so that (see (5.3))

$$(5.17)\quad 0 \le (1 - \widehat{h}(z)) \le \gamma, \qquad |z| \le \tau.$$

By considering separately the regions $s \le \tau/\epsilon$ and $s > \tau/\epsilon$ we see that

$$(5.18)\quad \begin{aligned} \frac{1 - \widehat{h}(\epsilon s)}{|s|^{d-(\sigma/2)}} &\le \gamma \frac{1}{|s|^{d-(\sigma/2)}} + \left(\frac{\epsilon}{\tau}\right)^\delta \frac{1}{|s|^{d-(\sigma/2)-\delta}} \\ &\le \gamma \left(\frac{1}{|s|^{d-(\sigma/2)}} + \frac{1}{|s|^{d-(\sigma/2)-\delta}} \right) \end{aligned}$$

for $\epsilon > 0$ sufficiently small. Here we can take any δ sufficiently small with $\sigma+\delta < \min(d, p\beta)$. Our Lemma then follows from Lemma 2.4 by first taking $\beta, \epsilon \to 0$ with $\gamma > 0$ fixed and then letting $\gamma \to 0$. □

Proof of Lemma 5.2 Define

(5.19)

$$\zeta_{\beta,\epsilon',\epsilon}\big([0,t_1]\times\cdots\times[0,t_p]\big) = \int_0^{t_1}\cdots\int_0^{t_p} \theta^2_{\beta,\epsilon'} * h_\epsilon(X_1(s_1)+\cdots+X_p(s_p))ds_1\cdots ds_p.$$

Lemma 5.2 will follow from the next two Lemmas.

Lemma 5.3

(5.20)

$$\limsup_{\epsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^p}\mathbb{E}\Big[(\zeta_{\beta,\epsilon',\epsilon}-\zeta_{\beta,\epsilon'})\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\Big]^n = -\infty.$$

Lemma 5.4

$$\limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^p}\mathbb{E}\Big[\zeta_{\beta,\epsilon',\epsilon}([0,\tau_1]\times\cdots\times[0,\tau_p])^n\Big] \le \log\rho. \tag{5.21}$$

Proof of Lemma 5.3 Following the same procedure used for (2.6),

$$\begin{aligned}
&\mathbb{E}\Big[(\zeta_{\beta,\epsilon'}-\zeta_{\beta,\epsilon',\epsilon})\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^n\Big] \qquad (5.22)\\
&= \int_{(\mathbb{R}^d)^n}\Big[\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n Q\Big(\sum_{j=1}^k\lambda_{\sigma(j)}\Big)\Big]^p\prod_{k=1}^n\big[(1-\widehat{h}(\epsilon\lambda))\wp_{\beta,\epsilon}*\wp_{\beta,\epsilon}(\lambda_k)\big]\,d\lambda_k\\
&\le \int_{(\mathbb{R}^d)^n}\Big[\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n Q\Big(\sum_{j=1}^k\lambda_{\sigma(j)}\Big)\Big]^p\prod_{k=1}^n\big[(1-\widehat{h}(\epsilon\lambda))\varphi_{d-\sigma}(\lambda_k)\big]\,d\lambda_k
\end{aligned}$$

by (5.6) and the proof follows as in the proof of Lemma 5.1. □

Proof of Lemma 5.4

Define

(5.23)

$$\zeta^z_{\beta,\epsilon}\big([0,t_1]\times\cdots\times[0,t_p]\big) = \int_0^{t_1}\cdots\int_0^{t_p}\theta^2_{\beta,\epsilon}(X_1(s_1)+\cdots+X_p(s_p)-z)ds_1\cdots ds_p.$$

Let $M > 0$ be fixed but arbitrary. By definition, using the fact that both $h_\epsilon(z)$ and $\zeta^z_{\beta,\epsilon'}$ are non-negative functions

$$
\begin{aligned}
(5.24)\qquad &\zeta_{\beta,\epsilon',\epsilon}\big([0,t_1]\times\cdots\times[0,t_p]\big)\\
&=\sum_{y\in\mathbb{Z}^d}\int_{[0,M]^d} h_\epsilon(yM+z)\zeta^{yM+z}_{\beta,\epsilon'}\big([0,t_1]\times\cdots\times[0,t_p]\big)dz\\
&\le\int_{[0,M]^d}\widetilde{h}_\epsilon(z)\widetilde{\zeta}^z_{\beta,\epsilon'}\big([0,t_1]\times\cdots\times[0,t_p]\big)dz
\end{aligned}
$$

where

(5.25)

$$
\widetilde{h}_\epsilon(x)=\sum_{y\in\mathbb{Z}^d}h_\epsilon(yM+z),\qquad \widetilde{\zeta}^z_{\beta,\epsilon'}\big([0,t_1]\times\cdots\times[0,t_p]\big)=\sum_{y\in\mathbb{Z}^d}\zeta^{yM+z}_{\beta,\epsilon'}\big([0,t_1]\times\cdots\times[0,t_p]\big)
$$

are two periodic functions on $\mathbb{R}^d$ with the period $M > 0$.

By Parseval's identity

$$
\begin{aligned}
(5.26)\qquad &\int_{[0,M]^d}\widetilde{h}_\epsilon(z)\widetilde{\zeta}^z_{\beta,\epsilon'}\big([0,t_1]\times\cdots\times[0,t_p]\big)dz\\
&=\frac{1}{M^d}\sum_{y\in\mathbb{Z}^d}\bigg(\int_{[0,M]^d}\widetilde{h}_\epsilon(x)\exp\Big\{-i\frac{2\pi}{M}(y\cdot x)\Big\}dx\bigg)\\
&\qquad\times\bigg(\int_{[0,M]^d}\widetilde{\zeta}^x_{\beta,\epsilon'}\big([0,t_1]\times\cdots\times[0,t_p]\big)\exp\Big\{i\frac{2\pi}{M}(y\cdot x)\Big\}dx\bigg).
\end{aligned}
$$

By periodicity

$$
\begin{aligned}
(5.27)\qquad &\int_{[0,M]^d}\widetilde{h}_\epsilon(x)\exp\Big\{-i\frac{2\pi}{M}(y\cdot x)\Big\}dx\\
&=\sum_{z\in\mathbb{Z}^d}\int_{[0,M]^d}h_\epsilon(zM+x)\exp\Big\{-i\frac{2\pi}{M}(y\cdot x)\Big\}dx\\
&=\sum_{z\in\mathbb{Z}^d}\int_{zM+[0,M]^d}h_\epsilon(x)\exp\Big\{-i\frac{2\pi}{M}\big(y\cdot(x-zM)\big)\Big\}dx\\
&=\sum_{z\in\mathbb{Z}^d}\int_{zM+[0,M]^d}h_\epsilon(x)\exp\Big\{-i\frac{2\pi}{M}(y\cdot x)\Big\}dx\\
&=\int_{\mathbb{R}^d}h_\epsilon(x)\exp\Big\{-i\frac{2\pi}{M}(y\cdot x)\Big\}dx=\widehat{h}\Big(\epsilon\frac{2\pi}{M}y\Big).
\end{aligned}
$$

Similarly, using (5.23)

$$\int_{[0,M]^d} \widetilde{\zeta}^x_{\beta,\epsilon'}\big([0,t_1]\times\cdots\times[0,t_p]\big)\exp\Big\{i\frac{2\pi}{M}(y\cdot x)\Big\}dx \tag{5.28}$$
$$= \int_{\mathbb{R}^d} \zeta^x_{\beta,\epsilon'}\big([0,t_1]\times\cdots\times[0,t_p]\big)\exp\Big\{i\frac{2\pi}{M}(y\cdot x)\Big\}dx$$
$$= \int_{[0,t_1]\times\cdots\times[0,t_p]} \wp_{\beta,\epsilon'} * \wp_{\beta,\epsilon'}(\frac{2\pi}{M}y)$$
$$\exp\Big\{i\frac{2\pi}{M}y\cdot\big(X_1(s_1)+\cdots+X_p(s_p)\big)\Big\}ds_1\cdots ds_p.$$

Hence,

$$\int_{[0,M]^d} \widetilde{h}_\epsilon(z)\widetilde{\zeta}^z_{\beta,\epsilon'}\big([0,t_1]\times\cdots\times[0,t_p]\big)dz \tag{5.29}$$
$$= \frac{1}{M^d}\sum_{y\in\mathbb{Z}^d}\widehat{h}\Big(\epsilon\frac{2\pi}{M}y\Big)\wp_{\beta,\epsilon'} * \wp_{\beta,\epsilon'}\Big(\frac{2\pi}{M}y\Big)$$
$$\int_{[0,t_1]\times\cdots\times[0,t_p]}\exp\Big\{i\frac{2\pi}{M}y\cdot\big(X_1(s_1)+\cdots+X_p(s_p)\big)\Big\}ds_1\cdots ds_p.$$

Using the same procedure as the one used to derive Lemma 2.2, (in fact, here we can proceed more directly, as in [2]) we can show that

$$\mathbb{E}\Big[\int_{[0,M]^d}\widetilde{h}_\epsilon(z)\widetilde{\zeta}^z_{\beta,\epsilon'}\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)dz\Big]^n \tag{5.30}$$
$$= \frac{1}{M^{dn}}\sum_{y_1,\cdots,y_n\in\mathbb{Z}^d}\bigg(\prod_{k=1}^n\widehat{h}\Big(\epsilon\frac{2\pi}{M}y_k\Big)\wp_{\beta,\epsilon'} * \wp_{\beta,\epsilon'}\Big(\frac{2\pi}{M}y_k\Big)\bigg)$$
$$\bigg[\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n Q\Big(\frac{2\pi}{M}\sum_{j=1}^k y_{\sigma(j)}\Big)\bigg]^p.$$

By [2, Theorem 4.1], (5.3), (5.6) and the fact that $\widehat{h}$ is supported in the set $[-2,2]^d$,

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^p}\mathbb{E}\Big[\int_{[0,M]^d}\widetilde{h}_\epsilon(z)\widetilde{\zeta}^z_{\beta,\epsilon'}\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)dz\Big]^n \tag{5.31}$$
$$=\log\Bigg(\frac{1}{M^d}\sup_{\|f\|_{2,\mathbb{Z}^d}=1}\sum_{x\in\mathbb{Z}^d}\widehat{h}\Big(\epsilon\frac{2\pi}{M}x\Big)\wp_{\beta,\epsilon'}*\wp_{\beta,\epsilon'}\Big(\frac{2\pi}{M}x\Big)$$
$$\Big[\sum_{y\in\mathbb{Z}^d}\sqrt{Q\Big(\frac{2\pi}{M}(x+y)\Big)}\sqrt{Q\Big(\frac{2\pi}{M}y\Big)}f(x+y)f(y)\Big]^p\Bigg)$$
$$\le\log\Big(M^{-d}\rho_M\Big)$$

where, setting $a=2\sqrt{d}/\epsilon$,

$$\rho_M=\sup_{\|f\|_{2,\mathbb{Z}^d}=1}\sum_{|x|\le(2\pi)^{-1}Ma}\wp_{\beta,0}*\wp_{\beta,0}\left(\frac{2\pi}{M}x\right)$$
$$\Big[\sum_{y\in\mathbb{Z}^d}\sqrt{Q\Big(\frac{2\pi}{M}(x+y)\Big)}\sqrt{Q\Big(\frac{2\pi}{M}y\Big)}f(x+y)f(y)\Big]^p. \tag{5.32}$$

In view of (5.24),

$$\limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^p}\mathbb{E}\Big[\zeta_{\beta,\epsilon',\epsilon}\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\Big]^n\le\log\Big(M^{-d}\rho_M\Big). \tag{5.33}$$

By Theorem 6.1 of the next section, letting $M\to\infty$ on the right hand side gives

$$\limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^p}\mathbb{E}\Big[\zeta_{\beta,\epsilon',\epsilon}\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\Big]^n\le\log\rho. \tag{5.34}$$

□

6 The limit as $M\to\infty$

Theorem 6.1 *Let ρ be defined in (1.4) and ρ_M be defined in (5.32). We have*

$$\limsup_{M\to\infty}M^{-d}\rho_M\le\rho. \tag{6.1}$$

Proof. For any $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, we write $[x] = ([x_1], \cdots, [x_d])$ for the lattice part of x (We also use the the notation $[\cdots]$ for parentheses without causing any confusion). For any $f \in \mathcal{L}^2(\mathbb{Z}^d)$ with $\|f\|_2 = 1$,

$$\begin{aligned}
&\sum_{|x|\le(2\pi)^{-1}Ma} \wp_{\beta,0} * \wp_{\beta,0}\left(\frac{2\pi}{M}x\right)\left[\sum_{y\in\mathbb{Z}^d}\sqrt{Q\Big(\frac{2\pi}{M}(x+y)\Big)}\sqrt{Q\Big(\frac{2\pi}{M}y\Big)}f(x+y)f(y)\right]^p \\
&= \int_{\{|\lambda|\le(2\pi)^{-1}Ma\}} \wp_{\beta,0} * \wp_{\beta,0}\left(\frac{2\pi}{M}[\lambda]\right) \\
&\qquad\qquad \left[\int_{\mathbb{R}^d}\sqrt{Q\Big(\frac{2\pi}{M}([\lambda]+[\gamma])\Big)}\sqrt{Q\Big(\frac{2\pi}{M}[\gamma]\Big)}f([\lambda]+[\gamma])f([\gamma])d\gamma\right]^p d\lambda \\
&= \left(\frac{M}{2\pi}\right)^d \int_{\{|\lambda|\le a\}} \wp_{\beta,0} * \wp_{\beta,0}\left(\frac{2\pi}{M}[\frac{M}{2\pi}\lambda]\right) \\
&\qquad\qquad \left[\left(\frac{M}{2\pi}\right)^d \int_{\mathbb{R}^d}\sqrt{Q_M\Big(\gamma+\frac{2\pi}{M}\Big[\frac{M}{2\pi}\lambda\Big]\Big)}\sqrt{Q_M(\gamma)}\right. \\
&\qquad\qquad\qquad \left.\times f\Big(\Big[\frac{M}{2\pi}\lambda\Big]+\Big[\frac{M}{2\pi}\gamma\Big]\Big)f\Big(\Big[\frac{M}{2\pi}\gamma\Big]\Big)d\gamma\right]^p d\lambda \qquad (6.2)
\end{aligned}$$

where

$$Q_M(\lambda) = Q\Big(\frac{2\pi}{M}\Big[\frac{M}{2\pi}\lambda\Big]\Big), \quad \lambda\in\mathbb{R}^d. \tag{6.3}$$

Write

$$g_0(\lambda) = \left(\frac{M}{2\pi}\right)^{d/2} f\Big(\Big[\frac{M}{2\pi}\lambda\Big]\Big), \quad \lambda\in\mathbb{R}^d. \tag{6.4}$$

We have

(6.5)

$$\int_{\mathbb{R}^d} g_0^2(\lambda)d\lambda = \left(\frac{M}{2\pi}\right)^d \int_{\mathbb{R}^d} f^2\Big(\Big[\frac{M}{2\pi}\lambda\Big]\Big)d\lambda = \int_{\mathbb{R}^d} f^2([\lambda])d\lambda = \sum_{x\in\mathbb{Z}^d} f^2(x) = 1.$$

We can also see that under this correspondence,

$$\left(\frac{M}{2\pi}\right)^{d/2} f\Big(\Big[\frac{M}{2\pi}\lambda\Big]+\Big[\frac{M}{2\pi}\gamma\Big]\Big) = g_0\Big(\gamma+\frac{2\pi}{M}\Big[\frac{M}{2\pi}\lambda\Big]\Big), \quad \lambda,\gamma\in\mathbb{R}^d. \tag{6.6}$$

Therefore, we need only to show that for any fixed $a > 0$

$$\limsup_{M\to\infty} \sup_{||g||_2=1} \int_{\{|\lambda|\le a\}} \wp_{\beta,0} * \wp_{\beta,0}\left(\frac{2\pi}{M}\Big[\frac{M}{2\pi}\lambda\Big]\right) \tag{6.7}$$
$$\bigg[\int_{\mathbb{R}^d} \sqrt{Q_M\Big(\gamma+\frac{2\pi}{M}\Big[\frac{M}{2\pi}\lambda\Big]\Big)}\sqrt{Q_M(\gamma)}g\Big(\gamma+\frac{2\pi}{M}\Big[\frac{M}{2\pi}\lambda\Big]\Big)g(\gamma)d\gamma\bigg]^p d\lambda$$
$$\le \sup_{||g||_2=1} \int_{\{|\lambda|\le a\}} \varphi_{d-\sigma}(\lambda)\bigg[\int_{\mathbb{R}^d} \sqrt{Q(\lambda+\gamma)}\sqrt{Q(\gamma)}g(\lambda+\gamma)g(\gamma)d\gamma\bigg]^p d\lambda.$$

To this end, note that by the inverse Fourier transformation the function

$$U_M(\lambda) = \int_{\mathbb{R}^d} \sqrt{Q_M(\gamma+\lambda)}\sqrt{Q_M(\gamma)}g(\gamma+\lambda)g(\gamma)d\gamma \tag{6.8}$$

is the Fourier transform of the function

$$\begin{aligned} V_M(x) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} U_M(\lambda)e^{-i\lambda\cdot x}d\lambda \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\lambda\cdot x}d\lambda \int_{\mathbb{R}^d} \sqrt{Q_M(\gamma+\lambda)}\sqrt{Q_M(\gamma)}g(\gamma+\lambda)g(\gamma)d\gamma \\ &= \frac{1}{(2\pi)^d}\iint_{\mathbb{R}^d\times\mathbb{R}^d} e^{-i(\lambda-\gamma)\cdot x}\sqrt{Q_M(\lambda)}g(\lambda)\sqrt{Q_M(\gamma)}g(\gamma)d\lambda d\gamma \\ &= \frac{1}{(2\pi)^d}\bigg|\int_{\mathbb{R}^d} e^{ix\cdot\gamma}\sqrt{Q_M(\gamma)}g(\gamma)d\gamma\bigg|^2. \end{aligned} \tag{6.9}$$

Therefore

$$\int_{\mathbb{R}^d} \sqrt{Q_M\Big(\gamma+\frac{2\pi}{M}\Big[\frac{M}{2\pi}\lambda\Big]\Big)}\sqrt{Q_M(\gamma)}g\Big(\gamma+\frac{2\pi}{M}\Big[\frac{M}{2\pi}\lambda\Big]\Big)g(\gamma)d\gamma$$
$$= U_M\Big(\frac{2\pi}{M}\Big[\frac{M}{2\pi}\lambda\Big]\Big)$$
$$= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \exp\Big\{ix\cdot\frac{2\pi}{M}\Big[\frac{M}{2\pi}\lambda\Big]\Big\}\bigg|\int_{\mathbb{R}^d} e^{ix\cdot\gamma}\sqrt{Q_M(\gamma)}g(\gamma)d\gamma\bigg|^2 dx$$
$$\le \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \Big|1-\exp\Big\{ix\cdot\Big(\lambda-\frac{2\pi}{M}\Big[\frac{M}{2\pi}\lambda\Big]\Big)\Big\}\Big|\cdot\bigg|\int_{\mathbb{R}^d} e^{ix\cdot\gamma}\sqrt{Q_M(\gamma)}g(\gamma)d\gamma\bigg|^2 dx$$
$$+\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\lambda}\bigg|\int_{\mathbb{R}^d} e^{ix\cdot\gamma}\sqrt{Q_M(\gamma)}g(\gamma)d\gamma\bigg|^2 dx. \tag{6.10}$$

By Parseval's identity and by the fact $Q_M \le 1$,

$$\begin{aligned} (6.11) \qquad & \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\bigg|\int_{\mathbb{R}^d} e^{ix\cdot\gamma}\sqrt{Q_M(\gamma)}g(\gamma)d\lambda\bigg|^2 dx \\ &= \int_{\mathbb{R}^d} Q_M(\gamma)g^2(\gamma)d\gamma \le \int_{\mathbb{R}^d} g^2(\gamma)d\gamma = 1. \end{aligned}$$

Hence, the first term on the right hand side of (6.10) tends to 0 uniformly over $\lambda \in \mathbb{R}^d$ and over all $g \in \mathcal{L}^2(\mathbb{R}^d)$ with $||g||_2 = 1$ as $M \to \infty$. The second term on the right hand side of (6.10) is equal to

(6.12)

$$\int_{\mathbb{R}^d} e^{ix\cdot\lambda}V_M(x)dx = U_M(\lambda) = \int_{\mathbb{R}^d} \sqrt{Q_M(\lambda+\gamma)}\sqrt{Q_M(\gamma)}g(\lambda+\gamma)g(\gamma)d\gamma.$$

Consequently, we will have (6.7) if we can prove

$$\begin{aligned} (6.13) \quad & \limsup_{M\to\infty}\sup_{||g||_2=1}\int_{\{|\lambda|\le a\}} \wp_{\beta,0} * \wp_{\beta,0}\Big(\frac{2\pi}{M}\Big[\frac{M}{2\pi}\lambda\Big]\Big) \\ & \qquad \times\bigg[\int_{\mathbb{R}^d}\sqrt{Q_M(\lambda+\gamma)}\sqrt{Q_M(\gamma)}g(\lambda+\gamma)g(\gamma)d\gamma\bigg]^p d\lambda \\ & \le \sup_{||g||_2=1}\int_{\{|\lambda|\le a\}} \varphi_{d-\sigma}(\lambda)\bigg[\int_{\mathbb{R}^d}\sqrt{Q(\lambda+\gamma)}\sqrt{Q(\gamma)}g(\lambda+\gamma)g(\gamma)d\gamma\bigg]^p d\lambda. \end{aligned}$$

By uniform continuity of the function Q we have that $Q_M(\cdot) \to Q(\cdot)$ uniformly on $\mathbb{R}^d$. Thus, given $\epsilon > 0$ we have

$$(6.14) \qquad \sup_{\lambda,\gamma\in\mathbb{R}^d}\Big|\sqrt{Q_M(\lambda+\gamma)}\sqrt{Q_M(\gamma)} - \sqrt{Q(\lambda+\gamma)}\sqrt{Q(\gamma)}\Big| < \epsilon.$$

for sufficiently large M. Therefore,

$$\begin{aligned} (6.15) \quad & \bigg\{\int_{\{|\lambda|\le a\}} d\lambda\bigg[\int_{\mathbb{R}^d}\sqrt{Q_M(\lambda+\gamma)}\sqrt{Q_M(\gamma)}g(\lambda+\gamma)g(\gamma)d\gamma\bigg]^p\bigg\}^{1/p} \\ & \le \epsilon\bigg\{\int_{\{|\lambda|\le a\}} d\lambda\bigg[\int_{\mathbb{R}^d} g(\lambda+\gamma)g(\gamma)d\gamma\bigg]^p\bigg\}^{1/p} \\ & + \bigg\{\int_{\{|\lambda|\le a\}} d\lambda\bigg[\int_{\mathbb{R}^d}\sqrt{Q(\lambda+\gamma)}\sqrt{Q(\gamma)}g(\lambda+\gamma)g(\gamma)d\gamma\bigg]^p\bigg\}^{1/p}. \end{aligned}$$

Also, since $\|g\|_2 = 1$,

$$\int_{\{|\lambda|\le a\}} d\lambda \left[\int_{\mathbb{R}^d} g(\lambda+\gamma) g(\gamma) d\gamma \right]^p \le C_d a^d \tag{6.16}$$

where C_d is the volume of a d-dimensional unit ball. (6.13) then follows using the uniform continuity of $\wp_{\beta,0} * \wp_{\beta,0}(\lambda)$, and finally (5.6). □

7 A variational formula

The goal of this section is to prove the Theorem 1.5. We begin with the following Lemma.

Lemma 7.1 *(1.19) holds under condition (1.4).*

Proof of Lemma 7.1: By (10.2), with σ replaced by $d-\sigma$

$$\begin{aligned} &\left(\int_{(\mathbb{R}^d)^p} \frac{\prod_{j=1}^p g^2(x_j)}{|x_1+\cdots+x_p|^\sigma} \prod_{j=1}^p dx_j \right)^{1/p} \\ &\qquad \le C\|g^2\|_{pd/(pd-\sigma)} = C\|g\|^2_{2pd/(pd-\sigma)}. \end{aligned} \tag{7.1}$$

We then use the fact that for some $c < \infty$

$$\|f\|_{2pd/(pd-\sigma)} \le c\|\widehat{f}\|_{2pd/(pd+\sigma)}, \qquad f \in \mathcal{S}(R^d), \tag{7.2}$$

and for any $r > 0$

$$\begin{aligned} &\|\widehat{f}\|^{2pd/(pd+\sigma)}_{2pd/(pd+\sigma)} \\ &= \int_{\mathbb{R}^d} \frac{(r+|\lambda|^\beta)^{pd/(pd+\sigma)}}{(r+|\lambda|^\beta)^{pd/(pd+\sigma)}} |\widehat{f}(\lambda)|^{2pd/(pd+\sigma)}\, d\lambda \\ &\le \|(r+|\lambda|^\beta)^{-pd/(pd+\sigma)}\|_{(pd+\sigma)/\sigma} \\ &\qquad \|(r+|\lambda|^\beta)^{pd/(pd+\sigma)} |\widehat{f}(\lambda)|^{2pd/(pd+\sigma)}\|_{(pd+\sigma)/pd}. \end{aligned} \tag{7.3}$$

Now if $\|f\|_2 = 1$ then

$$\begin{aligned} &\|(r+|\lambda|^\beta)^{pd/(pd+\sigma)} |\widehat{f}(\lambda)|^{2pd/(pd+\sigma)}\|_{(pd+\sigma)/pd} \\ &= (r+\mathcal{E}_\beta(f,f))^{pd/(pd+\sigma)} \end{aligned} \tag{7.4}$$

and

$$
\begin{aligned}
h_{p,r} &=: \|(r+|\lambda|^{\beta})^{-pd/(pd+\sigma)}\|_{(pd+\sigma)/\sigma} \\
&= \left(\int_{R^d} \frac{1}{(r+|\lambda|^{\beta})^{pd/\sigma}}\, d\lambda\right)^{\sigma/(pd+\sigma)}. \qquad (7.5)
\end{aligned}
$$

Since $p\beta > \sigma$ this is finite and $\lim_{r\to\infty} h_{p,r} = 0$. Together we have shown that

$$
\left(\int_{(\mathbb{R}^d)^p} \frac{\prod_{j=1}^p g^2(x_j)}{|x_1+\cdots+x_p|^{\sigma}} \prod_{j=1}^p dx_j\right)^{1/p} \leq c h_{p,r}^{(pd+\sigma)/pd}\left(r+\mathcal{E}_{\beta}(g,g)\right). \qquad (7.6)
$$

Our Lemma follows on taking r sufficiently large so that $c h_{p,r}^{(pd+\sigma)/pd} \leq 1$. □

Let $\mathcal{H}$ be a Hilbert space with norm $\|f\|$. We say that a (possibly unbounded) functional L on $\mathcal{H}$ is *positively homogeneous of order* k if for any $\lambda \in R^1$ and $f \in \mathcal{H}$

$$
L(\lambda f) = |\lambda|^k L(f). \qquad (7.7)
$$

The following simple Lemma will be very useful.

Lemma 7.2 *Let $L, \widetilde{L}$ be positive and positively homogeneous functionals on $\mathcal{H}$ of order 2. For any $\theta > 0$ let*

$$
\begin{aligned}
\Lambda(\theta) &= \sup_{\|f\|=1} \left(\theta L(f) - \widetilde{L}(f)\right) \qquad (7.8) \\
&= \sup_{f\in\mathcal{H}} \frac{\left(\theta L(f) - \widetilde{L}(f)\right)}{\|f\|^2}
\end{aligned}
$$

and assume that $\Lambda(\theta)$ is continuous. Let

$$
\begin{aligned}
J &= \sup_{\|f\|^2+\widetilde{L}(f)=1} L(f) \qquad (7.9) \\
&= \sup_{f\in\mathcal{H}} \frac{L(f)}{\|f\|^2+\widetilde{L}(f)}
\end{aligned}
$$

and assume that $J < \infty$. Then

$$
\Lambda\left(\frac{1}{J}\right) = 1. \qquad (7.10)
$$

Proof of Lemma 7.2: Fix $\epsilon > 0$ and choose $g \in \mathcal{H}$ with $\|g\|^2 + \widetilde{L}(g) = 1$ such that

$$L(g) \geq J - \epsilon. \tag{7.11}$$

Then

$$\begin{aligned} \Lambda\left(\frac{1}{(J-\epsilon)}\right) &\geq \frac{\left((J-\epsilon)^{-1}L(g) - \widetilde{L}(g)\right)}{\|g\|^2} \\ &\geq \frac{\left((J-\epsilon)^{-1}(J-\epsilon) - \widetilde{L}(g)\right)}{1 - \widetilde{L}(g)} = 1. \end{aligned} \tag{7.12}$$

By the continuity of $\Lambda(\theta)$, on taking $\epsilon \to 0$ we see that $\Lambda\left(\frac{1}{J}\right) \geq 1$.

On the other hand, by (7.9), for any $f \in \mathcal{H}$

$$L(f) \leq J\left(\|f\|^2 + \widetilde{L}(f)\right) \tag{7.13}$$

so that

$$\begin{aligned} \Lambda\left(\frac{1}{J}\right) &= \sup_{\|f\|=1}\left(J^{-1}L(f) - \widetilde{L}(f)\right) \\ &\leq \sup_{\|f\|=1}\left(J^{-1}J\left(\|f\|^2 + \widetilde{L}(f)\right) - \widetilde{L}(f)\right) = 1. \end{aligned} \tag{7.14}$$

□

Proof of Theorem 1.5

We take $\mathcal{H} = L^2(R^d,\, dx)$, $\widetilde{L}(f) = \mathcal{E}_\beta(f,f) = (2\pi)^{-d}\int |\widehat{f}(\lambda)|^2\psi(\lambda)\,d\lambda$ and

$$L(f) = \left(\int_{\mathbb{R}^{pd}} \frac{\prod_{j=1}^p |f(x_j)|^2}{|x_1 + \cdots + x_p|^\sigma} \prod_{j=1}^p dx_j\right)^{1/p}. \tag{7.15}$$

If $f_\epsilon(x) = \epsilon^{d/2} f(\epsilon x)$ then $L(f_\epsilon) = \epsilon^{\sigma/p}L(f)$, and $\widetilde{L}(f_\epsilon) = \epsilon^\beta \widetilde{L}(f)$. Thus

$$\begin{aligned} \Lambda(\theta) &= \sup_{\|f\|_2=1}\left(\theta L(f) - \widetilde{L}(f)\right) \\ &= \sup_{\|f\|_2=1}\left(\theta L(f_\epsilon) - \widetilde{L}(f_\epsilon)\right) \\ &= \sup_{\|f\|_2=1}\left(\theta \epsilon^{\sigma/p} L(f) - \epsilon^\beta \widetilde{L}(f)\right). \end{aligned} \tag{7.16}$$

Taking $\epsilon = \theta^{1/(\beta-\sigma/p)}$ we see that

$$\Lambda(\theta) = \theta^{\beta/(\beta-\sigma/p)}\Lambda(1) \tag{7.17}$$

which shows that $\Lambda(\theta)$ is continuous and that we can write (7.10) as

$$J = (\Lambda(1))^{1-\sigma/p\beta}. \tag{7.18}$$

Recall that

$$\rho = \sup_{\|f\|_2=1} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{f(\lambda+\gamma)f(\gamma)}{\sqrt{1+\psi(\lambda+\gamma)}\sqrt{1+\psi(\gamma)}} d\gamma \right]^p \varphi_{d-\sigma}(\lambda)d\lambda. \tag{7.19}$$

Setting $f = g/\sqrt{1+\psi}$ and using the notation $Q = (1+\psi)^{-1}$ we have that

$$\begin{aligned} \rho &= \sup_{\langle g,Qg\rangle=1} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} (Qg)(\lambda+\gamma)(Qg)(\gamma)d\gamma \right]^p \varphi_{d-\sigma}(\lambda)d\lambda \\ &= \sup_{\langle g,Qg\rangle=1} \int_{\mathbb{R}^d} \left[(Qg) * (\widetilde{Qg}) \right]^p (-\lambda)\varphi_{d-\sigma}(\lambda)d\lambda \end{aligned} \tag{7.20}$$

where $\widetilde{f}(\gamma) = f(-\gamma)$. Then, using $\mathcal{F}$ to denote the Fourier transform on $\mathbb{R}^d$, by Parseval's identity, which can be justified as in the proof of Lemma 2.2,

$$\rho = \sup_{\langle g,Qg\rangle=1} (2\pi)^{-d} \int_{\mathbb{R}^d} \mathcal{F}\left(\left[(Qg) * (\widetilde{Qg}) \right]^p \right)(x)\mathcal{F}\varphi_{d-\sigma}(x)dx. \tag{7.21}$$

Using the facts that $\mathcal{F}(f*g) = \mathcal{F}(f)\mathcal{F}(g)$, $\mathcal{F}(fg) = (2\pi)^{-d}\mathcal{F}(f)*\mathcal{F}(g)$ and (1.37), and using the notation f^{*p} for the p-fold convolution product of f with itself we see that

$$\begin{aligned} \rho &= \sup_{\langle g,Qg\rangle=1} (2\pi)^{-d(p+1)} \int_{\mathbb{R}^d} \left[|\widehat{Qg}|^2 \right]^{*p} (x) \frac{1}{|x|^\sigma} dx \\ &= \sup_{(2\pi)^d\|h\|_2^2 + (2\pi)^d \widetilde{L}(h)=1} (2\pi)^{d(p-1)} \int_{\mathbb{R}^d} |h^2|^{*p}(x) \frac{1}{|x|^\sigma} dx \end{aligned} \tag{7.22}$$

where in the last line we set $h = (2\pi)^{-d}\widehat{Qg}$ so that $\widetilde{g} = Q^{-1}\widehat{h}$ and therefore $(g, Qg) = (\widehat{h}, Q^{-1}\widehat{h}) = (\widehat{h}, (1+\psi)\widehat{h}) = (2\pi)^d\|h\|_2^2 + (2\pi)^d\widetilde{L}(h)$. By a change of variables we see that

$$\rho = \sup_{\|h\|_2^2 + \widetilde{L}(h)=1} (2\pi)^{-d} \int_{\mathbb{R}^{pd}} \frac{\prod_{j=1}^p |h(x_j)|^2}{|x_1 + \cdots + x_p|^\sigma} \prod_{j=1}^p dx_j = (2\pi)^{-d} J^p \tag{7.23}$$

and consequently by (7.18)

$$\rho = (2\pi)^{-d}(\Lambda(1))^{p-\sigma/\beta}. \tag{7.24}$$

□

8 Large deviations for $\zeta^*\big([0,1]^p\big)$

By Theorem 1.2, the non-trivial part of Theorem 1.3 is the upper bound.

Lemma 8.1 *For any $(t_1,\cdots,t_p)$, $M<\infty$ and any $\gamma>0$ sufficiently small so that $\sigma'=\sigma+\gamma$ satisfies (1.4), there is a $c=c(M,\delta)>0$ such that*

$$\sup_x \mathbb{E}\exp\left\{c\sup_{\substack{y\in B(x,M)\\ y\neq x}}\left(\frac{\big|(\zeta^y-\zeta^x)\big([0,t_1]\times\cdots\times[0,t_p]\big)\big|}{|y-x|^\gamma}\right)^{1/p}\right\}<\infty \tag{8.1}$$

and

$$\sup_x \mathbb{E}\exp\left\{c\sup_{\substack{y\in B(x,M)\\ y\neq x}}\left(\frac{\big|(\zeta^y-\zeta^x)\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\big|}{|y-x|^\gamma}\right)^{1/p}\right\}<\infty. \tag{8.2}$$

Proof of Lemma 8.1 By (2.15) there is a $C_0=C_0(\zeta,\psi,p)>0$ such that

$$\sup_{y\neq z}\mathbb{E}\left|\frac{(\zeta^y-\zeta^z)\big([0,t_1]\times\cdots\times[0,t_p]\big)}{|y-z|^\gamma}\right|^n\le (n!)^p C_0^n \quad n=0,1,2,\cdots. \tag{8.3}$$

Recall that a function Ψ: $\mathbb{R}^+\longrightarrow\mathbb{R}^+$ is called a *Young's function* if it is convex, increasing and satisfies $\Psi(0)=0$, $\lim_{x\to\infty}\Psi(x)=\infty$. The Orlicz space $\mathcal{L}_\Psi(\Omega,\mathcal{A},\mathbb{P})$ is defined as the linear space of all random variables X on the probability space $(\Omega,\mathcal{A},\mathbb{P})$ such that

$$||X||_\Psi=\inf\big\{c>0;\ \ \mathbb{E}\Psi(c^{-1}|X|)\le 1\big\}. \tag{8.4}$$

It is known that $||\cdot||_\Psi$ defines a norm (called the Orlicz norm) and $\mathcal{L}_\Psi(\Omega,\mathcal{A},\mathbb{P})$ becomes a Banach space under $||\cdot||_\Psi$.

We now choose the Young function Ψ such that $\Psi(x) = \rho(x) - p^{-1}\exp(\frac{p-1}{p})$ where $\rho(x) = \exp\left\{x^{1/p}\right\}$ when $|x| \geq (\frac{p-1}{p})^p$, and for $0 \leq |x| \leq (\frac{p-1}{p})^p$ we define $\rho(x)$ as the tangent line to $y = \exp\left\{x^{1/p}\right\}$ at the point $((\frac{p-1}{p})^p, \exp(\frac{p-1}{p}))$. (This complicated definition is due to the fact that $\rho(x) = \exp\left\{x^{1/p}\right\}$ is only convex for $|x| \geq (\frac{p-1}{p})^p$). By (8.3) there is $c = c(\zeta, d, p) > 0$ such that

$$||(\zeta^y - \zeta^z)\big([0,t_1] \times \cdots \times [0,t_p]\big)||_\Psi \leq c|y-z|^\gamma, \qquad \forall y, z. \tag{8.5}$$

By a standard chaining argument (see, e.g., Lemma 9 in [3]), for any $\gamma' < \gamma$, $M < \infty$, uniformly in x

$$\Bigg|\Bigg| \sup_{\substack{y \in B(x,M)\\ y \neq x}} \frac{\big|(\zeta^y - \zeta^x)\big([0,t_1] \times \cdots \times [0,t_p]\big)\big|}{|y-x|^{\gamma'}} \Bigg|\Bigg|_\Psi < \infty \tag{8.6}$$

which leads to (8.1), after renaming γ' as γ. The proof of (8.2) is similar, as one can easily see that (2.15) holds with all t_i replaced by τ_i. □

Now choosing γ so that (8.1) holds, pick λ so that $(1+\gamma\lambda)/p = \beta/\sigma$. By (1.4) we have that $\lambda > 0$. It then follows from (8.1) that for some $C < \infty$ and all $t \geq 1$

$$\begin{aligned} &\sup_{x \in \mathbb{R}^d} \mathbb{P}\Big\{ \sup_{y \in B(x,\epsilon t^{-\lambda})} \big|\zeta^x([0,1]^p) - \zeta^y([0,1]^p)\big| \geq \delta t \Big\} \\ &\leq \sup_{x \in \mathbb{R}^d} \mathbb{P}\Big\{ \sup_{y \in B(x,\epsilon t^{-\lambda})} \frac{\big|\zeta^x([0,1]^p) - \zeta^y([0,1]^p)\big|}{|x-y|^\gamma} \geq \frac{\delta t}{\epsilon^\gamma t^{-\gamma\lambda}} \Big\} \\ &\leq C e^{-\left(\frac{\delta t}{\epsilon^\gamma t^{-\gamma\lambda}}\right)^{1/p}} = C e^{-\left(\frac{\delta}{\epsilon^\gamma}\right)^{1/p} t^{(1+\gamma\lambda)/p}} = C e^{-\left(\frac{\delta}{\epsilon^\gamma}\right)^{1/p} t^{\beta/\sigma}}. \end{aligned} \tag{8.7}$$

Consequently,

$$\begin{aligned} &\lim_{\epsilon \to 0^+} \limsup_{t \to \infty} t^{-\beta/\sigma} \log \sup_{x \in \mathbb{R}^d} \\ &\qquad \mathbb{P}\Big\{ \sup_{y \in B(x,\epsilon t^{-\lambda})} \big|\zeta^x([0,1]^p) - \zeta^y([0,1]^p)\big| \geq \delta t \Big\} = -\infty. \end{aligned} \tag{8.8}$$

We first consider the case of $\beta = 2$, the case of Brownian motion. By (8.8), for some $\lambda > 0$ we have that for any $\delta > 0$

$$\begin{aligned} &\lim_{\epsilon \to 0^+} \limsup_{t \to \infty} t^{-2/\sigma} \log \sup_{x \in \mathbb{R}^d} \\ &\qquad \mathbb{P}\Big\{ \sup_{y \in B(x,\epsilon t^{-\lambda})} \big|\zeta^x([0,1]^p) - \zeta^y([0,1]^p)\big| \geq \delta t \Big\} = -\infty. \end{aligned} \tag{8.9}$$

Since the supremum of a continuous Gaussian process has Gaussian tails, see for example [19, Corollary 5.4.6], we have

(8.10)
$$\lim_{M\to\infty}\limsup_{t\to\infty} t^{-2/\sigma}\log\mathbb{P}\Big\{\sup_{s_1,\cdots,s_p\le 1}|X_1(s_1)+\cdots+X_p(s_p)|\ge Mt^{1/\sigma}\Big\}=-\infty.$$

When $\sup_{s_1,\cdots,s_p\le 1}|X_1(s_1)+\cdots+X_p(s_p)|\le Mt^{1/\sigma}$ and $|x|\ge 2Mt^{1/\sigma}$,

(8.11)
$$\zeta^x([0,1]^p)\le t.$$

Thus

(8.12)
$$\begin{aligned}&\mathbb{P}\Big\{\sup_{x\in\mathbb{R}^d}\zeta^x([0,1]^p)\ge t\Big\}\\ &\le\mathbb{P}\Big\{\sup_{|x|\le 2Mt^{1/\sigma}}\zeta^x([0,1]^p)\ge t\Big\}\\ &+\mathbb{P}\Big\{\sup_{s_1,\cdots,s_p\le 1}|X_1(s_1)+\cdots+X_p(s_p)|\ge Mt^{1/\sigma}\Big\}.\end{aligned}$$

The cardinality of an $\epsilon t^{-\lambda}$-net on the ball of radius $2Mt^{1/\sigma}$ is of the order $O(t^{d(\lambda+\sigma^{-1})})$. This gives

(8.13)
$$\begin{aligned}&\mathbb{P}\Big\{\sup_{|x|\le 2Mt^{1/\sigma}}\zeta^x([0,1]^p)\ge t\Big\}\\ &\le Ct^{d(\lambda+\sigma^{-1})}\left\{\sup_{x\in\mathbb{R}^d}\mathbb{P}\Big\{\zeta^x([0,1]^p)\ge(1-\delta)t\Big\}\right.\\ &\left.+\sup_{x\in\mathbb{R}^d}\mathbb{P}\Big\{\sup_{y\in B(x,\epsilon t^{-\lambda})}\Big|\zeta^x([0,1]^p)-\zeta^y([0,1]^p)\Big|\ge\delta t\Big\}\right\}.\end{aligned}$$

Therefore,

(8.14)
$$\limsup_{t\to\infty}t^{-2/\sigma}\log\mathbb{P}\Big\{\sup_{x\in\mathbb{R}^d}\zeta^x([0,1]^p)\ge t\Big\}\le\max\{a,b,c\}$$

(8.15)

where
$$a=\limsup_{t\to\infty}t^{-2/\sigma}\log\sup_{x\in\mathbb{R}^d}P\Big\{\zeta^x([0,1]^p)\ge(1-\delta)t\Big\},$$

$$b = \limsup_{t\to\infty} t^{-2/\sigma} \log \sup_{x\in\mathbb{R}^d} P\Big\{ \sup_{y\in B(x,\epsilon t^{-\lambda})} \big|\zeta^x([0,1]^p) - \zeta^y([0,1]^p)\big| \geq \delta t\Big\},$$

$$c = \limsup_{t\to\infty} t^{-2/\sigma} \log P\Big\{ \sup_{s_1,\cdots,s_p\leq 1} |X_1(s_1) + \cdots + X_p(s_p)| \geq Mt^{1/\sigma}\Big\}\Big\}.$$

Letting $M \to \infty$ and then $\epsilon \to 0$ on the right-hand side, by (8.9) and (8.10) we have

$$\begin{aligned}(8.16)\qquad &\limsup_{t\to\infty} t^{-2/\sigma} \log \mathbb{P}\Big\{ \sup_{x\in\mathbb{R}^d} \zeta^x([0,1]^p) \geq t\Big\} \\ &\leq \limsup_{t\to\infty} t^{-2/\sigma} \log \sup_{x\in\mathbb{R}^d} \mathbb{P}\Big\{\zeta^x([0,1]^p) \geq (1-\delta)t\Big\}.\end{aligned}$$

Using (4.34) and (2.25)

$$(8.17)\quad \limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{(n!)^{\sigma/\beta}} \sup_{x\in\mathbb{R}^d} \mathbb{E}\Big[\zeta^x([0,1]^p)^n\Big] \leq \log\Big(\frac{\beta p}{\beta p - \sigma}\Big)^{\frac{\beta p-\sigma}{\beta}} + \log\rho.$$

The (easy part of the) proof of [17, Lemma 2.3] then shows that

$$(8.18)\quad \limsup_{t\to\infty} t^{-\beta/\sigma} \log \sup_{x\in\mathbb{R}^d} \mathbb{P}\Big\{\zeta^x([0,1]^p) \geq t\Big\} \leq -\frac{\sigma}{\beta}\Big(\frac{p\beta-\sigma}{p\beta}\Big)^{\frac{p\beta-\sigma}{\sigma}} \rho^{-\beta/\sigma}.$$

With $\beta = 2$ we have

$$\begin{aligned}(8.19)\qquad &\limsup_{t\to\infty} t^{-2/\sigma} \log \sup_{x\in\mathbb{R}^d} \mathbb{P}\Big\{\zeta^x([0,1]^p) \geq (1-\delta)t\Big\} \\ &\qquad\leq -(1-\delta)^{2/\sigma}\frac{\sigma}{2}\Big(\frac{2p-\sigma}{2p}\Big)^{\frac{2p-\sigma}{\sigma}} \rho^{-2/\sigma}.\end{aligned}$$

Thus

$$\begin{aligned}(8.20)\qquad &\limsup_{t\to\infty} t^{-2/\sigma} \log \mathbb{P}\Big\{ \sup_{x\in\mathbb{R}^d} \zeta^x([0,1]^p) \geq t\Big\} \\ &\qquad\leq -(1-\delta)^{2/\sigma}\frac{\sigma}{2}\Big(\frac{2p-\sigma}{2p}\Big)^{\frac{2p-\sigma}{\sigma}} \rho^{-2/\sigma}.\end{aligned}$$

Letting $\delta \to 0^+$ gives (1.12).

We now consider $\beta \neq 2$. We will show that there exists $c_1 > 0$ such that

$$(8.21)\qquad \mathbb{E}\Big(\exp\Big(c_1\Big\{\sup_{z\in\mathbb{R}^d} \zeta^x\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\Big\}^{1/p}\Big)\Big) < \infty.$$

It will follow from this that for some $c_2 < \infty$

$$\mathbb{E}\Big(\Big\{\sup_{z\in\mathbb{R}^d}\zeta^x\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\Big\}^{n/p}\Big) \le n!c_2^n \tag{8.22}$$

for all n. Hence, taking $n = mp$

$$\mathbb{E}\Big(\Big\{\sup_{z\in\mathbb{R}^d}\zeta^x\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\Big\}^{m}\Big) \le (mp)!c_2^{pm} \le (m!)^p c_3^m. \tag{8.23}$$

Using (2.24) and Stirling's formula as in (2.25) we obtain

$$\limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^{\sigma/\beta}}\mathbb{E}\Big[\zeta^*([0,1]^p)^n\Big] \le c_4 < \infty. \tag{8.24}$$

Then once again the (easy part of the) proof of [17, Lemma 2.3] will show that for some $0 < C < \infty$

$$\limsup_{t\to\infty} t^{-\beta/\sigma}\log P\Big\{\zeta^*([0,1]^p) \ge t\Big\} \le -C. \tag{8.25}$$

Thus it only remains to show (8.21).

Lemma 8.2 *Let X_t be a d-dimensional symmetric stable process of order β and τ an independent exponential of parameter 1. Then there exists a constant c_1 such that for $D > 0$,*

$$\mathbb{P}(\sup_{s\le\tau}|X_s| \ge D) \le \frac{c_1}{D^\beta}. \tag{8.26}$$

Proof. It is well known, [16, Proposition 2.2], that the density of X_t satisfies

$$p(t,x,y) \le ct/|x-y|^{d+\beta}.$$

(A better estimate is possible for larger t, but this is not needed.) Integrating over $|y-x| \ge D$, we obtain

$$\mathbb{P}^x(|X_t - X_0| \ge D) \le \frac{ct}{D^\beta}. \tag{8.27}$$

We now obtain an estimate on the exit probabilities. Let $S = \inf\{s : |X_s| \ge D\}$. If $\sup_{s\le t} |X_s| \ge D$, then $S \le t$ and either $|X_t| \ge D/2$ or $|X_t| \le D/2$, so that $|X_S - X_t| \ge D/2$. Thus

$$\mathbb{P}(\sup_{s\le t} |X_s| \ge D) \le \mathbb{P}(|X_t| \ge D/2) \\ + \mathbb{P}(S < t, |X_t - X_S| \ge D/2).$$

The first term on the right is bounded by ct/D^β using (8.27). The second term on the right is bounded by

$$\int_0^t \mathbb{P}(|X_t - X_s| \ge D/2)\,\mathbb{P}(S \in ds) \le 2c \int_0^t (t-s)/D^\beta \mathbb{P}(S \in ds) \le 2ct/D^\beta$$

using (8.27) again and the Markov property of X.

Finally,

$$\mathbb{P}(\sup_{s\le\tau} |X_s| \ge D) = \int_0^\infty e^{-t}\mathbb{P}(\sup_{s\le t} |X_s| \ge D)\,dt \le \int_0^\infty e^{-t}\frac{ct}{D^\beta}\,dt \\ \le c/D^\beta$$

as desired. □

Lemma 8.3 *Suppose for each $z \in \mathbb{R}^d$ there is a random variable Y^z such that $z \to Y^z$ is continuous, a.s., and there exist δ, A and B such that*

(8.28) $$\mathbb{E}e^{A|Y^z|} \le B, \qquad z \in R^d,$$

(8.29) $$\mathbb{E}e^{A|Y^z - Y^{z'}|/|z-z'|^\delta} \le B, \qquad z, z' \in R^d.$$

Then there exist c_1 and c_2 such that for every $M \ge 1$

(8.30) $$\mathbb{E}\exp\Big(c_1 A \sup_{|z|\le M} |Y^z|\Big) \le c_2 M^{2d} B.$$

This follows by a standard chaining argument; we give the proof for the sake of completeness.

Proof. Let $Q_k = B(0,M)\cap 2^{-k}Z^d$ and $Q=\cup_k Q_k$. Since $z\to Y^z$ is continuous, it suffices to bound

$$\mathbb{E}\exp\Big(c_1 A\sup_{|z|\in Q}|Y^z|\Big). \tag{8.31}$$

If $z\in Q$, we write

$$z=z_0+(z_1-z_0)+(z_2-z_1)+\cdots .$$

Here z_i is the point of Q_i closest to z, with some convention for breaking ties. Since $z\in Q_k$ for some k, the above sum is actually a finite one.

If $|Y^z|\ge\lambda$, then either the event R holds: $|Y^{z_0}|\ge\lambda/2$ for some $z_0\in Q_0$, or for some i the event S_i holds: $|Y^{z_{i+1}}-Y^{z_i}|\ge\lambda/20i^2$ for some pair z_i,z_{i+1} with $z_i\in Q_i$, $z_{i+1}\in Q_{i+1}$, and $|z_i-z_{i+1}|\le\sqrt{d}2^{-i}$.

Since there are at most M^d points in Q_0, using (8.28) we see the probability of the event R is bounded by

$$cBM^de^{-A\lambda/2}.$$

For each i, there are at most $cM^{2d}2^{c'id}$ pairs z_i,z_{i+1} as in the definition of the even S_i, so the probability of the event S_i is bounded by

$$cBM^{2d}2^{c'id}\sup_{|z_i-z_{i+1}|\le\sqrt{d}2^{-i}}\mathbb{P}\Big(\frac{|Y^{z_{i+1}}-Y^{z_i}|}{|z_{i+1}-z_i|^\delta}\ge\frac{\lambda A}{20i^2(\sqrt{d}2^{-i})^\delta}\Big) \tag{8.32}$$

$$\le cBM^{2d}2^{c'id}\exp\Big(-c''\frac{\lambda A}{20i^2(\sqrt{d}2^{-i})^\delta}\Big). \tag{8.33}$$

If we sum over i the probabilities of the events S_i holding and add to that the probability of the event R holding, we obtain

$$\mathbb{P}(\sup_{z\in Q}|Y^z|\ge\lambda)\le cBM^{2d}e^{-c'\lambda A}.$$

Our result follows from this. □

We now prove (8.21). Let X^i_t, $i=1,\dots,p$, be independent d-dimensional symmetric stable processes of order β. We write simply ζ^z for $\zeta^z([0,\tau_1]\times\cdots\times[0,\tau_p])$ and Z_i for $\sup_{s\le\tau_i}|X^i_s|$. We will choose c_1 later.

It follows from (2.6) and (2.16) that there exists c_2 such that

$$\sup_{z\in\mathbb{R}^d} \mathbb{E} \exp\Big(c_2|\zeta^z|^{1/p}\Big) < \infty, \tag{8.34}$$

and using (8.2) and the fact that $|a^{1/p} - b^{1/p}| \le |a-b|^{1/p}$, we can choose c_2 such that also

$$\sup_{z,z'\in\mathbb{R}^d} \mathbb{E} \exp\Big(c_2|\,|\zeta^z|^{1/p} - |\zeta^{z'}|^{1/p}|/|z-z'|^\delta\Big) < \infty. \tag{8.35}$$

Write

$$\mathbb{E}e^{c_1 \sup_z |\zeta^z|^{1/p}} = \mathbb{E}\Big[e^{c_1 \sup_z |\zeta^z|^{1/p}}; \max_{1\le i\le p} Z^i \le 1\Big] \tag{8.36}$$

$$+ \sum_{k=0}^\infty \mathbb{E}\Big[e^{c_1 \sup_z |\zeta^z|^{1/p}}; 2^k \le \max_{1\le i\le p} Z_i \le 2^{k+1}\Big] \tag{8.37}$$

$$:= I + \sum_{k=0}^\infty J_k. \tag{8.38}$$

Now write

$$I \le \mathbb{E}\Big[e^{c_1 \sup_{|z|\le 2p} |\zeta^z|^{1/p}}\Big] + \mathbb{E}\Big[e^{c_1 \sup_{|z|>2p} |\zeta^z|^{1/p}}; \max_{1\le i\le p} Z \le 1\Big] = I' + I''. \tag{8.39}$$

Provided $c_1 < c_2$, then I' is finite by Lemma 8.3 with $Y^z = |\zeta^z|^{1/p}$. If $|z| \ge 2p$ and $\max_{1\le i\le p} Z_i \le 1$, then

$$|Z_1 + \cdots + Z_p - z| \ge p,$$

and hence

$$I'' \le \mathbb{E}e^{c_1 p^{-\delta/p}(\tau_1\cdots\tau_p)^{1/p}} = \int_{\mathbb{R}_+^p} e^{-\sum_{j=1}^p t_j} e^{c_1 p^{-\delta/p}(t_1\cdots t_p)^{1/p}}\, dt_1 \cdots dt_p.$$

Since $(t_1\cdots t_p)^{1/p} \le \max_{1\le j\le p} t_j \le \sum_{j=1}^p t_j$ we see that

$$\mathbb{E}e^{c(\tau_1\cdots\tau_p)^{1/p}} < \infty \tag{8.40}$$

if c is small enough.

Combining with the estimate for I' shows that I is finite, provided $c_1 < c_2$ and c_2 is sufficiently small.

We turn to J_k and write

$$\begin{aligned}
J_k &\le \mathbb{E}\Big[e^{c_1 \sup_{|z|\le p2^{k+1}} |\zeta^z|^{1/p}}; 2^k \le \max_{1\le i\le p} Z_i \le 2^{k+1}\Big] \\
&\quad + \mathbb{E}\Big[e^{c_1 \sup_{|z|> p2^{k+1}} |\zeta^z|^{1/p}}; 2^k \le \max_{1\le i\le p} Z_i \le 2^{k+1}\Big] \\
&= J_k' + J_k''.
\end{aligned}$$

For J_k' we apply Hölder's inequality with $\frac{1}{r}+\frac{1}{s}$ and r and s to be chosen later. Then

$$\begin{aligned}
J_k' &\le \Big(\mathbb{E} e^{c_1 r \sup_{|z|\le p2^{k+1}} |\zeta^z|^{1/p}}\Big)^{1/r} \Big(\mathbb{P}(\max_{1\le i\le p} Z_i \ge 2^k)\Big)^{1/s} \\
&\le \Big(c2^{2dk}\Big)^{1/r} \Big(\frac{c}{2^{k\beta}}\Big)^{1/s}
\end{aligned}$$

by Lemma 8.3. We now choose r and s so that $\eta := \beta/s - 2d/r > 0$, and hence $2^{k\beta/s} \ge 2^{\eta k} 2^{2dk/r}$. This proves J_k' is summable in k.

To handle J_k'', if $\max_{1\le i\le p} Z_i \le 2^{k+1}$ and $|z| \ge p2^{k+2}$, then

$$\sup_{s_1\le 1,\dots,s_p\le 1} |X^1_{s_1} + \cdots + X^p_{s_p} - z| \ge p2^{k+1},$$

and hence

$$J_k'' \le \mathbb{E}\Big[e^{c_1 (p2^{k+1})^{-\sigma/p} (\tau_1\cdots\tau_p)^{1/p}}; \max_{1\le i\le p} Z_i \ge 2^k\Big].$$

Using Cauchy-Schwarz, we obtain

$$J_k'' \le e^{c_1 (p2^{k+1})^{-\sigma/p}}\ \mathbb{P}(\max_{1\le i\le p} Z_i \ge 2^k).$$

The second factor is less than or equal to $c/2^{k\beta}$, which is summable in k.

Finally we choose c_1 small enough that $c_1 r < c_2$, and the proof of (8.21) is complete. □

9 Laws of the iterated logarithm

The upper bounds in (1.15) and (1.16) and therfore the upper bound in (1.14) follows from Theorem 1.2, the scaling property given in (1.8), and a standard procedure using the Borel-Cantelli lemma. It remains to prove that

$$\limsup_{t\to\infty} t^{-\frac{p\beta-\sigma}{\beta}}(\log\log t)^{-\sigma/\beta}\zeta([0,t]^p) \ge \left(\frac{\sigma}{\beta}\right)^{-\sigma/\beta}\left(\frac{p\beta-\sigma}{p\beta}\right)^{\frac{\sigma-p\beta}{\beta}}\rho \tag{9.1}$$

almost surely.

We first prove that

$$\lim_{\delta\to 0^+}\liminf_{t\to\infty} t^{-1}\log\mathbb{P}\Big\{\inf_{|y|\le\delta}\zeta^y\big([0,t]^p\big)\ge t^p\Big\} \ge -\frac{\sigma}{\beta}\Big(\frac{p\beta-\sigma}{p\beta}\Big)^{\frac{p\beta-\sigma}{\sigma}}\rho^{-\beta/\sigma}. \tag{9.2}$$

Using (8.2) and Chebyshev's inequality we have that for any $\epsilon>0$,

$$\limsup_{\delta\to 0^+}\limsup_{t\to\infty} t^{-1}\log\mathbb{P}\Big\{\sup_{|y|\le\delta}\big|(\zeta^0-\zeta^y)\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\big|\ge\epsilon t^p\Big\}=-\infty. \tag{9.3}$$

On the other hand,

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{|y|\le\delta}\big|(\zeta^0-\zeta^y)\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\big|\ge\epsilon t^p\Big\} \\
&=\int_0^\infty\cdots\int_0^\infty e^{-(t_1+\cdots+t_p)} \\
&\qquad\qquad \mathbb{P}\Big\{\sup_{|y|\le\delta}\big|(\zeta^0-\zeta^y)\big([0,t_1]\times\cdots\times[0,t_p]\big)\big|\ge\epsilon t^p\Big\}dt_1\cdots dt_p \\
&\ge\int_{(1-\epsilon)t}^t\cdots\int_{(1-\epsilon)t}^t e^{-(t_1+\cdots+t_p)} \\
&\qquad\qquad \mathbb{P}\Big\{\sup_{|y|\le\delta}\big|(\zeta^0-\zeta^y)\big([0,t_1]\times\cdots\times[0,t_p]\big)\big|\ge\epsilon t^p\Big\}dt_1\cdots dt_p \\
&\ge\big(e^{-(1-\epsilon)t}-e^{-t}\big)^p \\
&\qquad \inf_{(1-\epsilon)t\le t_1,\cdots,t_p\le t}\mathbb{P}\Big\{\sup_{|y|\le\delta}\big|(\zeta^0-\zeta^y)\big([0,t_1]\times\cdots\times[0,t_p]\big)\big|\ge\epsilon t^p\Big\}.
\end{aligned} \tag{9.4}$$

So we have

$$\limsup_{\delta\to 0^+}\limsup_{t\to\infty} t^{-1}\log \inf_{(1-\epsilon)t\le t_1,\cdots,t_p\le t} \mathbb{P}\Big\{\sup_{|y|\le\delta}\big|(\zeta^0-\zeta^y)\big([0,t_1]\times\cdots\times[0,t_p]\big)\big|\ge \epsilon t^p\Big\} = -\infty. \tag{9.5}$$

For any t and $(1-\epsilon)t\le t_1,\cdots,t_p\le t$,

$$\begin{aligned}\inf_{|y|\le\delta}\zeta^y\big([0,t]^p\big) &\ge \inf_{|y|\le\delta}\zeta^y\big([0,t_1]\times\cdots\times[0,t_p]\big)\\ &\ge \zeta^0\big([0,t_1]\times\cdots\times[0,t_p]\big)-\sup_{|y|\le\delta}\big|(\zeta^0-\zeta^y)\big([0,t_1]\times\cdots\times[0,t_p]\big)\big|\\ &\ge \zeta^0\big([0,(1-\epsilon)t]^p\big)-\sup_{|y|\le\delta}\big|(\zeta^0-\zeta^y)\big([0,t_1]\times\cdots\times[0,t_p]\big)\big|.\end{aligned} \tag{9.6}$$

Hence,

$$\begin{aligned}&\mathbb{P}\Big\{\inf_{|x|\le\delta}\zeta^x\big([0,t]^p\big)\ge t^p\Big\}\\ &+\inf_{(1-\epsilon)t\le t_1,\cdots,t_p\le t}\mathbb{P}\Big\{\sup_{|x|\le\delta}\big|(\zeta^0-\zeta^x)\big([0,t_1]\times\cdots\times[0,t_p]\big)\big|\ge\epsilon t^p\Big\}\\ &\ge \mathbb{P}\Big\{\zeta^0\big([0,(1-\epsilon)t]^p\big)\ge(1+\epsilon)t^p\Big\}.\end{aligned} \tag{9.7}$$

Consequently,

$$\begin{aligned}&\max\bigg\{\liminf_{t\to\infty}t^{-1}\log\mathbb{P}\Big\{\inf_{|y|\le\delta}\zeta^y\big([0,t]^p\big)\ge t^p\Big\},\\ &\qquad\limsup_{t\to\infty}t^{-1}\log\inf_{(1-\epsilon)t\le t_1,\cdots,t_p\le t}\\ &\qquad\qquad\mathbb{P}\Big\{\sup_{|y|\le\delta}\big|(\zeta^0-\zeta^y)\big([0,t_1]\times\cdots\times[0,t_p]\big)\big|\ge\epsilon t^p\Big\}\bigg\}\\ &\ge\liminf_{t\to\infty}t^{-1}\log\mathbb{P}\Big\{\zeta^0\big([0,(1-\epsilon)t]^p\big)\ge(1+\epsilon)t^p\Big\}.\end{aligned} \tag{9.8}$$

Notice that by the scaling (1.8)

$$\mathbb{P}\Big\{\zeta^0\big([0,(1-\epsilon)t]^p\big)\ge(1+\epsilon)t^p\Big\}=\mathbb{P}\Big\{\zeta^0\big([0,1]^p\big)\ge(1+\epsilon)(1-\epsilon)^{-\frac{\beta p-\sigma}{\beta}}t^{\sigma/\beta}\Big\}, \tag{9.9}$$

so that by Theorem 1.2,

$$\lim_{t\to\infty} t^{-1}\log\mathbb{P}\Big\{\zeta^0\big([0,(1-\epsilon)t]^p\big)\ge(1+\epsilon)t^p\Big\} = -(1+\epsilon)^{\beta/\sigma}(1-\epsilon)^{-\frac{p\beta-\sigma}{\sigma}}\frac{\sigma}{\beta}\Big(\frac{p\beta-\sigma}{p\beta}\Big)^{\frac{p\beta-\sigma}{\sigma}}\rho^{-\beta/\sigma}. \tag{9.10}$$

Let $\delta\to 0^+$ in (9.8). By (9.8), (9.5) and (9.10) we obtain

$$\lim_{\delta\to 0^+}\liminf_{t\to\infty} t^{-1}\log\mathbb{P}\Big\{\inf_{|y|\le\delta}\zeta^y\big([0,t]^p\big)\ge t^p\Big\} \ge -(1+\epsilon)^{\beta/\sigma}(1-\epsilon)^{-\frac{p\beta-\sigma}{\sigma}}\frac{\sigma}{\beta}\Big(\frac{p\beta-\sigma}{p\beta}\Big)^{\frac{p\beta-\sigma}{\sigma}}\rho^{-\beta/\sigma}. \tag{9.11}$$

Letting $\epsilon\to 0^+$ on the right hand side leads to (9.2).

We now come to the proof of (9.1). For each $k\ge 1$, write $t_k=k^k$ and define

$$X_{j,k}(t)=X_j(t_k+t)-X_j(t_k)\quad t\ge 0,\;\; j=1,\cdots,p,\;\; k=1,2,\cdots. \tag{9.12}$$

Let $\zeta_k^x([a,b]^p)$ be the Riesz potential of the additive stable process

$$\overline{X}_k(s_1,\cdots,s_p)=X_{1,k}(s_1)+\cdots+X_{p,k}(s_p). \tag{9.13}$$

Then for each k, $\{\zeta_k^x\,, x\in\mathbb{R}^d\}\overset{d}{=}\{\zeta^x\,, x\in\mathbb{R}^d\}$. Let $\delta>0$ be a small number which will be specified later. Write $Y_k=X_1(t_k)+\cdots+X_p(t_k)$. A rough estimate gives that with probability 1, the inequality

$$|Y_k|\le 2^{-1}\delta\Big(\frac{t_{k+1}}{\log\log t_{k+1}}\Big)^{1/\beta} \tag{9.14}$$

eventually holds. Therefore

$$\zeta\big([t_k,t_{k+1}]^p\big)=\zeta_k^{Y_k}\big([0,t_{k+1}-t_k]^p\big) \ge \inf_{|y|\le\delta(t_{k+1}/\log\log t_{k+1})^{1/\beta}}\zeta_k^y\big([0,t_{k+1}-t_k]^p\big) \tag{9.15}$$

eventually holds, almost surely. For each k, by the scaling (1.8),

$$(9.16)\qquad \inf_{|y|\le\delta(t_{k+1}/\log\log t_{k+1})^{1/\beta}}\zeta_k^y\big([0,t_{k+1}-t_k]^p\big)$$
$$\overset{d}{=}\inf_{|y|\le\delta(t_{k+1}/\log\log t_{k+1})^{1/\beta}}\zeta^y\big([0,t_{k+1}-t_k]^p\big)$$
$$\overset{d}{=}\Big(\frac{t_{k+1}}{\log\log t_{k+1}}\Big)^{\frac{\beta p-\sigma}{\beta}}\inf_{|y|\le\delta}\zeta^y\big([0,\ t_{k+1}^{-1}(t_{k+1}-t_k)\log\log t_{k+1}]^p\big).$$

Let $\theta>0$ satisfy

$$(9.17)\qquad \theta<\Big(\frac{\beta}{\sigma}\Big)^{\frac{\sigma}{\beta}}\Big(1-\frac{\sigma}{\beta p}\Big)^{-(p-\frac{\sigma}{\beta})}\rho.$$

We have

$$(9.18)\quad \mathbb{P}\Big\{\inf_{|y|\le\delta(t_{k+1}/\log\log t_{k+1})^{1/\beta}}\zeta_k^y\big([0,t_{k+1}-t_k]^p\big)\ge\theta t_{k+1}^{\frac{\beta p-\sigma}{\beta}}(\log\log t_{k+1})^{\frac{\sigma}{\beta}}\Big\}$$
$$=\mathbb{P}\Big\{\inf_{|y|\le\delta}\zeta^y\big([0,\ t_{k+1}^{-1}(t_{k+1}-t_k)\log\log t_{k+1}]^p\big)\ge\theta(\log\log t_{k+1})^p\Big\}.$$

Using the scaling (1.8) once again

$$(9.19)\quad \mathbb{P}\Big\{\inf_{|y|\le\delta}\zeta^y\big([0,\ t_{k+1}^{-1}(t_{k+1}-t_k)\log\log t_{k+1}]^p\big)\ge\theta(\log\log t_{k+1})^p\Big\}$$
$$=\mathbb{P}\Big\{\inf_{|y|\le\delta\theta^{1/\sigma}}\zeta^y\big([0,\ t_{k+1}^{-1}(t_{k+1}-t_k)\theta^{\beta/\sigma}\log\log t_{k+1}]^p\big)\ge(\theta^{\beta/\sigma}\log\log t_{k+1})^p\Big\}.$$

By (9.2), therefore, one can take $\delta>0$ sufficiently small so that

$$(9.20)\quad \liminf_{k\to\infty}\frac{1}{\log\log t_{k+1}}\log\mathbb{P}\Big\{\inf_{|y|\le\delta(t_{k+1}/\log\log t_{k+1})^{1/\beta}}\zeta_k^y\big([0,t_{k+1}-t_k]^p\big)$$
$$\ge\theta t_{k+1}^{\frac{\beta p-\sigma}{\beta}}(\log\log t_{k+1})^{\frac{\sigma}{\beta}}\Big\}>-1.$$

Consequently,

$$(9.21)\quad \sum_k\mathbb{P}\Big\{\inf_{|y|\le\delta(t_{k+1}/\log\log t_{k+1})^{1/\beta}}\zeta_k^y\big([0,t_{k+1}-t_k]^p\big)\ge\theta t_{k+1}^{\frac{\beta p-\sigma}{\beta}}(\log\log t_{k+1})^{\frac{\sigma}{\beta}}\Big\}=\infty.$$

Notice that

$$(9.22)\qquad \inf_{|y|\le\delta(t_{k+1}/\log\log t_{k+1})^{1/\beta}}\zeta_k^y\big([0,t_{k+1}-t_k]^p\big)\qquad k=1,2,\cdots$$

is an independent sequence. By the Borel-Cantelli lemma,

(9.23)

$$\limsup_{k\to\infty} t_{k+1}^{-\frac{\beta p-\sigma}{\beta}}(\log\log t_{k+1})^{-\frac{\sigma}{\beta}} \inf_{|y|\le\delta(t_{k+1}/\log\log t_{k+1})^{1/\beta}} \zeta_k^y\big([0, t_{k+1}-t_k]^p\big) \ge \theta \quad a.s.$$

By (9.15),

$$\limsup_{k\to\infty} t_{k+1}^{-\frac{\beta p-\sigma}{\beta}}(\log\log t_{k+1})^{-\frac{\sigma}{\beta}}\zeta\big([t_k, t_{k+1}]^p\big) \ge \theta \quad a.s. \tag{9.24}$$

Consequently,

$$\limsup_{t\to\infty} t^{-\frac{\beta p-\sigma}{\beta}}(\log\log t)^{-\frac{\sigma}{\beta}}\zeta\big([0,t]^p\big) \ge \theta \quad a.s. \tag{9.25}$$

Letting

$$\theta \uparrow \Big(\frac{\beta}{\sigma}\Big)^{\frac{\sigma}{\beta}}\Big(1-\frac{\sigma}{\beta p}\Big)^{-(p-\frac{\sigma}{\beta})}\rho \tag{9.26}$$

proves (9.1). □

10 Appendix: Sobolev-type inequalities

Lemma 10.1 *For any $q>1$ and integer $p\ge 1$*

$$\|f_1 * \cdots * f_p\|_q \le C^p \prod_{l=1}^p \|f_l\|_{pq/((p-1)q+1)} \tag{10.1}$$

and for any $0<\sigma<d$

$$\left|\int_{(\mathbb{R}^d)^p} \frac{\prod_{l=1}^p f_l(x_l)}{|x_1+\cdots+x_p|^{d-\sigma}}\prod_{j=1}^p dx_l\right| \le C^p\prod_{l=1}^p \|f_l\|_{pd/((p-1)d+\sigma)}. \tag{10.2}$$

Furthermore, for any n and any $F_l = F_l(x_{l,j};\, 1\le j\le n)$, $1\le l\le p$ we have

$$\left|\int_{(\mathbb{R}^d)^{np}} \prod_{j=1}^n \frac{1}{|x_{1,j}+\cdots+x_{p,j}|^{d-\sigma}} \prod_{l=1}^p F_l \prod_{j=1}^n\prod_{l=1}^p dx_{l,j}\right| \le C^p \prod_{l=1}^p \|F_l\|_{pd/((p-1)d+\sigma)}, \tag{10.3}$$

and more generally, for some $C < \infty$ independent of $z \in R^d$

$$\left| \int_{(\mathbb{R}^d)^{np}} \prod_{j=1}^{n} \frac{1}{|x_{1,j} + \cdots + x_{p,j} - z|^{d-\sigma}} \; \prod_{l=1}^{p} F_l \; \prod_{j=1}^{n} \prod_{l=1}^{p} dx_{l,j} \right| \le C^p \prod_{l=1}^{p} \|F_l\|_{pd/((p-1)d+\sigma)}. \tag{10.4}$$

Proof: We prove (10.1) by induction on p. The case $p = 1$ is trivial. Thus assume (10.1) holds for all $p \le m - 1$. Since $t^{-1} = r^{-1} + s^{-1} - 1$ when $t = q, r = mq/((m-1)q + 1), s = mq/(m - 1 + q)$, it follows from Young's inequality, [5], p. 275, that

$$\|f_1 * \cdots * f_m\|_q \le C \|f_1\|_{mq/((m-1)q+1)} \|f_2 * \cdots * f_m\|_{mq/(m-1+q)}. \tag{10.5}$$

By our induction hypothesis and using the fact that

$$\frac{(m-1)mq}{(m-2)mq + m - 1 + q} = \frac{(m-1)mq}{(m-1)^2 q + m - 1} = \frac{mq}{(m-1)q + 1} \tag{10.6}$$

we see that

$$\|f_2 * \cdots * f_m\|_{mq/(m-1+q)} \le C^{m-1} \prod_{l=2}^{m} \|f_l\|_{mq/((m-1)q+1)} \tag{10.7}$$

which completes the proof of (10.1).

To prove (10.2) we write

$$\int_{(\mathbb{R}^d)^p} \frac{\prod_{l=1}^p f_l(x_l)}{|x_1 + \cdots + x_p|^{d-\sigma}} \prod_{l=1}^{p} dx_l = \int_{(\mathbb{R}^d)^2} \frac{f_1(x) \; (f_2 * \cdot * f_p)(y)}{|x - y|^{d-\sigma}} \, dx \, dy \tag{10.8}$$

and apply (1.26) with $r = pd/((p-1)d + \sigma)$ so that $s = pd/((p-1)\sigma + d)$ to obtain

$$\left| \int_{(\mathbb{R}^d)^p} \frac{\prod_{l=1}^p f_l(x_l)}{|x_1 + \cdots + x_p|^{d-\sigma}} \prod_{l=1}^{p} dx_l \right| \le C \|f_1\|_{pd/((p-1)d+\sigma)} \; \|f_2 * \cdot * f_p\|_{pd/((p-1)\sigma+d)}. \tag{10.9}$$

Then using (10.1) and the fact that

$$\frac{(p-1)pd}{(p-2)pd+(p-1)\sigma+d} = \frac{(p-1)pd}{(p-1)^2d+(p-1)\sigma} = \frac{pd}{(p-1)d+\sigma} \tag{10.10}$$

we obtain (10.2).

We next prove (10.3). By (10.2)

$$\begin{aligned}
&\left| \int_{(\mathbb{R}^d)^{np}} \prod_{j=1}^{n} \frac{1}{|x_{1,j}+\cdots+x_{p,j}|^{d-\sigma}} \ \prod_{l=1}^{p} F_l \prod_{j=1}^{n}\prod_{l=1}^{p} dx_{l,j} \right| \\
&\leq \int_{(\mathbb{R}^d)^{(n-1)p}} \prod_{j=2}^{n} \frac{1}{|x_{1,j}+\cdots+x_{p,j}|^{d-\sigma}} \\
&\qquad \left| \int_{(\mathbb{R}^d)^{p}} \frac{\prod_{l=1}^{p} |F_l|}{|x_{1,1}+\cdots+x_{p,1}|^{d-\sigma}} \prod_{l=1}^{p} dx_{l,1} \right| \prod_{j=2}^{n}\prod_{l=1}^{p} dx_{l,j} \\
&\leq \int_{(\mathbb{R}^d)^{(n-1)p}} \prod_{j=2}^{n} \frac{1}{|x_{1,j}+\cdots+x_{p,j}|^{d-\sigma}} \\
&\qquad\qquad \prod_{l=1}^{p} \|F_l\|_{pd/((p-1)d+\sigma), x_{l,1}} \prod_{j=2}^{n}\prod_{l=1}^{p} dx_{l,j}
\end{aligned} \tag{10.11}$$

where

$$\|F_l\|_{q,\, x_{l,1}} = \left(\int_{\mathbb{R}^{dp}} |F_l(x_{l,j};\ 1 \leq j \leq n)|^q \, dx_{l,1} \right)^{1/q}.$$

Inequality (10.3) then follows by iterating this step. For example, the next iteration will bound (10.11) by

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^{(n-2)p}} \prod_{j=3}^{n} \frac{1}{|x_{1,j}+\cdots+x_{p,j}|^{d-\sigma}} \\
&\qquad\qquad \prod_{l=1}^{p} \|F_l\|_{pd/((p-1)d+\sigma), x_{l,1}, x_{l,2}} \prod_{j=3}^{n}\prod_{l=1}^{p} dx_{l,j}
\end{aligned} \tag{10.12}$$

where now

$$\begin{aligned}(10.13)\qquad &\|F_l\|_{q,x_{l,1},x_{l,2}}\\ &=\left(\int_{\mathbb{R}^d}\|F_l\|^q_{q,\,x_{l,1}}\,dx_{l,2}\right)^{1/q}\\ &=\left(\int_{\mathbb{R}^{2d}}|F_l(x_{l,j};\,1\le j\le n)|^q\,dx_{l,1}\,dx_{l,2}\right)^{1/q}.\end{aligned}$$

It should be clear that this will lead to (10.3).

Now let $T_z^{l,j}$ denote translation of $x_{l,j}$ by z and set $\mathcal{T}=\prod_{j=1}^n T_z^{1,j}$. Then using (10.3) and the translation invariance of Lebesgue measure

$$\begin{aligned}(10.14)\qquad &\left|\int_{(\mathbb{R}^d)^{np}}\prod_{j=1}^n\frac{1}{|x_{1,j}+\cdots+x_{p,j}-z|^{d-\sigma}}\,\prod_{l=1}^p F_l\,\prod_{j=1}^n\prod_{l=1}^p dx_{l,j}\right|\\ &=\left|\int_{(\mathbb{R}^d)^{np}}\prod_{j=1}^n\frac{1}{|x_{1,j}+\cdots+x_{p,j}|^{d-\sigma}}\,\prod_{l=1}^p \mathcal{T}F_l\,\prod_{j=1}^n\prod_{l=1}^p dx_{l,j}\right|\\ &\le C^p\prod_{l=1}^p\|\mathcal{T}F_l\|_{pd/((p-1)d+\sigma)}=C^p\prod_{l=1}^p\|F_l\|_{pd/((p-1)d+\sigma)}\end{aligned}$$

which is (10.4). □

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