

Divergence form operators on fractal-like domains

Martin T. Barlow¹
Department of Mathematics
University of British Columbia
Vancouver, V6T 1Z2
Canada

and

Richard F. Bass²
Department of Mathematics
University of Connecticut
Storrs, CT 06269
U.S.A.

Abstract. We consider elliptic operators \mathcal{L} in divergence form on certain domains in \mathbb{R}^d with fractal volume growth. The domains we look at are pre-Sierpinski carpets, which are derived from higher dimensional Sierpinski carpets. We prove a Harnack inequality for non-negative \mathcal{L} -harmonic functions on these domains and establish upper and lower bounds for the corresponding heat equation.

Keywords. Harnack inequality, divergence form, Sierpinski carpet, fractals, heat equation, elliptic operators, fundamental solution, diffusions

AMS Subject Classifications. Primary: 35J15; Secondary: 60J60, 60J35

¹ Research partially supported by a NSERC (Canada) grant.

² Research partially supported by NSF Grant DMS 9700721.

1. Introduction.

In recent years there has been interest in the connection between differential inequalities, such as Poincaré inequalities and Sobolev inequalities, and global properties of the heat kernel, such as a parabolic Harnack inequality or pointwise bounds. The spaces considered can be manifolds, graphs, or domains in \mathbb{R}^d . For work in this area see, for example, [Gr], [D], [AC], [St].

To give a flavor of what is known, let M be a non-compact complete connected Riemannian manifold, with bounded geometry. Let Δ be the Laplace-Beltrami operator on M , $d(x, y)$ the Riemannian metric, $B(x, r)$ the ball of radius r centered at x , $V(x, r)$ the volume of $B(x, r)$, and $p(t, x, y)$ the heat kernel on M . Then the following are equivalent (see [Gr], [SC]):

(A1) M satisfies the Poincaré inequality

$$\inf_a \int_{B(x,r)} |f - a|^2 \leq c_1 r^2 \int_{B(x,r)} |\nabla f|^2$$

and a volume doubling condition.

(A2) The parabolic Harnack inequality holds on M for solutions to the heat equation $\partial u / \partial t = \Delta u$.

(A3) The heat kernel $p(t, x, y)$ satisfies the bounds

$$c_2 V(x, t^{1/2}) e^{-c_3 d(x,y)^2/t} \leq p(t, x, y) \leq c_4 V(x, t^{1/2}) e^{-c_5 d(x,y)^2/t}.$$

See also [D] for a similar statement for graphs. The hardest part of this equivalence is the derivation of the parabolic Harnack inequality from (A1). The argument used in [Gr] and [SC] is a generalization of Moser's argument [M1], [M2].

This equivalence is intimately related to Moser's parabolic Harnack inequality for solutions to the heat equation

$$\frac{\partial u}{\partial t} = \mathcal{L}u,$$

where \mathcal{L} is a uniformly elliptic operator in divergence form given by

$$\mathcal{L}f(x) = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} f \right) (x)$$

and the domain is \mathbb{R}^d . Recall that the Laplace-Beltrami operator for a manifold is given in local coordinates by a scalar function times an operator of the form \mathcal{L} . All proofs of Moser's parabolic Harnack inequality use in an essential way the fact that the energy forms for the

Laplacian and for \mathcal{L} , given by $\mathcal{E}_\Delta(f, f) = \int_{\mathbb{R}^d} |\nabla f|^2$ and $\mathcal{E}_\mathcal{L}(f, f) = \int_{\mathbb{R}^d} \nabla f \cdot a \nabla f$, respectively, are comparable.

A parallel set of activity has been the investigation of heat kernel bounds on fractal sets. These sets exhibit two differences from the cases discussed above:

- (B1) Since the local structure is non-Euclidean, it is necessary to do some work to prove the existence of a suitable ‘Laplacian’ operator.
- (B2) These sets exhibit different space-time scaling properties than \mathbb{R}^d .

While (B1) may suggest that the results on heat kernels on fractals are special to that context, this is not the case. Given any regular fractal F it is possible to define a graph, manifold, or domain in \mathbb{R}^d with a global structure which mimics the local structure of F . Following [O] we will call these ‘pre-fractals’. The regularity of F means that the volume doubling condition holds for these spaces. One expects that the long time behavior of the heat kernel on a pre-fractal will be similar to the short-time behavior on the fractal F , and this has been verified in certain cases – see [J], [BB4].

Since a pre-fractal manifold is a manifold, any general analysis of heat kernels on manifolds has to include this case. Further, pre-fractal manifolds may provide useful examples of certain extremal situations – see [BCG].

For a sufficiently regular pre-fractal, such as the pre-carpet considered in [BB3], a Poincaré inequality holds and takes the form

$$\inf_a \int_{B(x,r)} |f - a|^2 \leq cr^{d_w} \int_{B(x,r)} |\nabla f|^2, \quad r \geq 1. \quad (1.1)$$

Here $d_w \geq 2$ is a constant (called the ‘walk dimension’) which governs the long range space-time scaling of the space. For \mathbb{R}^d one has $d_w = 2$. One explanation for the anomalous scaling in (1.1) can be given by reformulating the inequality in variational terms: consider the problem of maximizing

$$\int_B |f|^2 \quad \text{subject to} \quad \int_B f = 0, \quad \int_{B(x,r)} |\nabla f|^2 = 1.$$

The holes in the pre-fractal allow one to make $|f|$ large without paying a big penalty in terms of energy.

The results in [BB3] are based on an elliptic Harnack inequality for Δ -harmonic functions satisfying Neumann boundary conditions, which was proved by a probabilistic argument and which used very strongly the symmetry of the space and the operator Δ . The standard parabolic Harnack inequality contains explicit information on the space-time scaling of the heat kernel, and it is shown in [BB3] that it fails for pre-carpet whenever $d_w \neq 2$. However,

it is proved in [BB3, Sect. 7] that a modified parabolic Harnack inequality holds for solutions to the heat equation on pre-carpets.

It is natural to ask whether these results can be extended to uniformly elliptic divergence form operators. However, the standard techniques used to prove Harnack inequalities, such as Moser iteration [M1], or the Nash-Davies-Fabes-Stroock method [FS] both encounter difficulties in the pre-fractal context. This is essentially because of the ‘anomalous’ scaling in the Poincaré inequality (1.1), which means that ‘minimum energy’ functions are very far from being approximately linear.

In this paper we consider one particular class of pre-fractals, domains in \mathbb{R}^d which are pre-Sierpinski carpets, and do prove a parabolic Harnack inequality of the Moser type. We expect that these methods will prove useful in investigating the connection between Poincaré and Harnack inequalities for more general spaces of pre-fractal type.

We also obtain estimates on the fundamental solution of the corresponding heat equation. The bounds are quite different from those for the heat equation on \mathbb{R}^d . In the latter case, Aronson’s bounds tell us that the fundamental solution $p(t, x, y)$ to $\partial u / \partial t = \mathcal{L}u$ on \mathbb{R}^d is comparable to

$$c_1 t^{-d/2} \exp(-c_2 |x - y|^2 / t).$$

More precisely, there exist constants c_1, c_2, c_3, c_4 such that $p(t, x, y)$ is bounded above by $c_1 t^{-d/2} \exp(-c_2 |x - y|^2 / t)$ and bounded below by $c_3 t^{-d/2} \exp(-c_4 |x - y|^2 / t)$ for all x, y, t . In contrast, for the heat equation on a pre-Sierpinski carpet, there exist constants $d_w > 2$ and d_s depending on the pre-Sierpinski carpet such that for all $t \geq 1$ such that $|x - y| \leq t$ the fundamental solution $q(t, x, y)$ is comparable to

$$c_1 t^{-d_s/2} \exp\left(-c_2 \left(\frac{|x - y|^{d_w}}{t}\right)^{1/(d_w - 1)}\right). \quad (1.2)$$

Sierpinski carpets are the fractal subsets of \mathbb{R}^d formed by the following procedure. Let $d \geq 2$ and let $F_0 = [0, 1]^d$. Let $l_{\mathcal{F}} \geq 3$ be an integer and divide F_0 into $(l_{\mathcal{F}})^d$ equal subcubes. Next remove a symmetric pattern of subcubes from F_0 and call what remains F_1 . Now repeat the procedure: divide each subcube that is contained in F_1 into $l_{\mathcal{F}}^d$ equal parts, remove the same symmetric pattern from each as was done to obtain F_1 from F_0 , and call what remains F_2 . Continuing in this way we obtain a decreasing sequence of (closed) subsets of $[0, 1]^d$. Let $\mathcal{F} = \bigcap_{n=0}^{\infty} F_n$; we call \mathcal{F} a *Sierpinski carpet* or simply, a *carpet*. The standard Sierpinski carpet (see [Sie]) is the carpet for which $d = 2$, $l_{\mathcal{F}} = 3$, and F_1 consists of F_0 minus the central square. If $d = 3$, $l_{\mathcal{F}} = 3$, and F_1 consists of F_0 minus the 7 subcubes that do not share an edge with F_0 , we obtain the Menger sponge; see [Man], p. 145 for a picture.

The domains we will consider are what are known as *pre-carpets* – see [O]. These are the sets $\mathcal{P} = \bigcup_{n=0}^{\infty} l_{\mathcal{F}}^n F_n$. (Here and throughout this paper we write $\lambda G = \{\lambda x : x \in G\}$). Note

that $\mathcal{P} \subset \mathbb{R}_+^d$, and that $\mathcal{P} \cap [0, l_{\mathcal{F}}^n]^d$ consists of $[0, l_{\mathcal{F}}^n]^d$ with a number of (possibly adjacent) cubical holes removed, of sides varying from 1 to $l_{\mathcal{F}}^{n-1}$. If Γ is the interior of \mathcal{P} , then Γ is a (non-empty) domain in \mathbb{R}^d with a piecewise linear boundary. We may regard pre-carpets as idealized models of a region with obstacles of many different sizes. The set \mathcal{P} is not a fractal, since the interior of \mathcal{P} is a non-empty domain in \mathbb{R}^d . However, if we write $V(x, R)$ for the volume of the intersection of \mathcal{P} with the Euclidean ball of radius R centered at x , then \mathcal{P} has ‘fractal volume growth’ in the sense that there exists $\alpha \in (1, d)$ such that

$$c_1 R^\alpha \leq V(x, R) \leq c_2 R^\alpha, \quad x \in \mathcal{P}, R \geq 1.$$

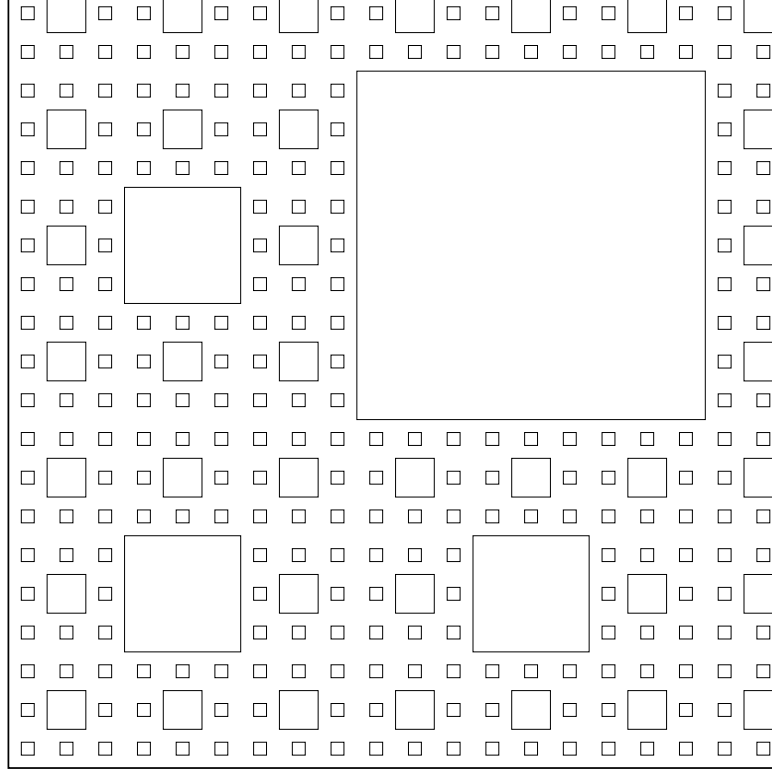


Figure 1: (Part of) a pre-carpet. The small squares have side length 1.

Let \mathcal{L} be the divergence form operator

$$\mathcal{L}f(x) = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} f \right) (x),$$

where the matrix $a_{ij}(x)$ is bounded, measurable, and symmetric for each x , and satisfies the uniform ellipticity condition

$$\lambda_1^2 |\xi|^2 \leq \sum_{i,j=1}^d \xi_i a_{ij}(x) \xi_j \leq \lambda_2^2 |\xi|^2, \quad x \in \mathcal{P}, \quad (1.3)$$

with $0 < \lambda_1 < \lambda_2 < \infty$. We will assume the a_{ij} are smooth, but our estimates will not depend on the smoothness of the a_{ij} . A function f is \mathcal{L} -harmonic on a subdomain D of \mathcal{P} if $\mathcal{L}f = 0$ there and the conormal derivative of f is 0 almost everywhere on $D \cap \partial\mathcal{P}$. For further information on diffusions with conormal reflection, see [PW].

The main result of this paper is the following elliptic Harnack inequality for \mathcal{L} -harmonic functions. For $x, y \in \mathcal{P}$ write $\gamma(x, y)$ for the (Euclidean) length of the shortest path in \mathcal{P} connecting x and y , and let $B_\gamma(x, r) = \{y \in \mathcal{P} : \gamma(x, y) < r\}$.

Theorem 1.1. *Let $x \in \mathcal{P}$, $R > 0$ and suppose f is nonnegative and \mathcal{L} -harmonic in $B_\gamma(x, 2R)$. There exists c_1 , not depending on x, R or f , such that*

$$f(x) \leq c_1 f(y) \quad \text{for } x, y \in B_\gamma(x, R). \quad (1.4)$$

A crucial point is that c_1 does not depend on R for otherwise this result is an easy consequence of the standard Moser's Harnack inequality. In a previous paper [BB3] we proved Theorem 1.1 in the case $\mathcal{L} = \frac{1}{2}\Delta$, using a probabilistic coupling argument. This Harnack inequality was then the key step in obtaining bounds on the fundamental solution to the heat equation on \mathcal{P} :

$$\frac{\partial u}{\partial t} = \Delta u, \quad (1.5)$$

where u has Neumann boundary conditions on $\partial\mathcal{P}$.

The arguments here do not replace those in [BB3]: we need various properties of the solutions to (1.5) (and the diffusion process W associated with them) to prove Theorem 1.1.

Our proof of Theorem 1.1 uses a variation of the Moser technique. The main difficulty is in finding suitable 'cut-off' functions. The classical arguments rely on the existence of suitable 'cut-off' functions with bounded gradient. The results of Kusuoka [K] for the Sierpinski gasket suggest that such functions do not exist in the carpet case, and that, while they do exist in the pre-carpet case, scaling of the right order will not hold. The key step is to prove a weighted Sobolev inequality, where the L^p norm is with respect to an energy measure for the pre-carpet, rather than Lebesgue measure. This then allows us to use cut-off functions derived from the potentials associated with the operator Δ on \mathcal{P} . We prove this Sobolev inequality by first using Dirichlet form techniques to obtain a weighted Poincaré inequality, then to derive a weighted Nash inequality, and finally from this we obtain the weighted Sobolev inequality.

As in [BB3], once we have an elliptic Harnack inequality it is the solutions of the associated heat equation $\partial u/\partial t = \mathcal{L}u$ on \mathcal{P} . See Theorem 5.3 for a precise statement.

The layout of this paper is as follows. Section 2 introduces the notation we will use together with a few basic facts. Section 3 contains the proof of the weighted Sobolev inequality. We prove Theorem 1.1 in Section 4: the key iteration argument is given in Proposition 4.2. The heat kernel bounds are derived in Section 5.

2. Notation and preliminaries.

We begin by setting up our notation. We use the letter c with subscripts to denote constants which depend only on the dimension d and the carpet \mathcal{F} . We renumber the constants for each lemma, proposition, theorem, and corollary. We use the notation $A \asymp B$ to mean $c_1 A \leq B \leq c_2 A$, where c_i are as above.

Let $d \geq 2$, $F_0 = [0, 1]^d$, and let $l_{\mathcal{F}} \in \mathbb{N}$, $l_{\mathcal{F}} \geq 3$ be fixed. For $n \in \mathbb{Z}$ let \mathcal{S}_n be the collection of closed cubes of side length $l_{\mathcal{F}}^{-n}$ with vertices in $l_{\mathcal{F}}^{-n} \mathbb{Z}^d$. For $A \subseteq \mathbb{R}^d$, set

$$\mathcal{S}_n(A) = \{S : S \subset A, S \in \mathcal{S}_n\}.$$

For $S \in \mathcal{S}_n$, let Ψ_S be the orientation preserving affine map which maps F_0 onto S .

We now define a decreasing sequence (F_n) of closed subsets of F_0 . Let $1 \leq m_{\mathcal{F}} < l_{\mathcal{F}}^d$ be an integer, and let F_1 be the union of $m_{\mathcal{F}}$ distinct elements of $\mathcal{S}_1(F_0)$. We impose the following conditions on F_1 :

Hypotheses 2.1.

(H1) (*Symmetry*) F_1 is preserved by all the isometries of the unit cube F_0 .

(H2) (*Connectedness*) The interior of F_1 is connected, and contains a path connecting the hyperplanes $\{x_1 = 0\}$ and $\{x_1 = 1\}$.

(H3) (*Non-diagonality*) Let B be a cube in F_0 which is the union of 2^d distinct elements of \mathcal{S}_1 . (So B has side length $2l_{\mathcal{F}}^{-1}$). Then if the interior of $F_1 \cap B$ is non-empty, it is connected.

(H4) (*Borders included*) F_1 contains the line segment $\{x : 0 \leq x_1 \leq 1, x_2 = \dots = x_d = 0\}$.

Of these, (H1) and (H2) are essential, while (H3) and (H4) could be weakened somewhat. See the discussion in [BB3].

We may think of F_1 as being derived from F_0 by removing the interiors of $l_{\mathcal{F}}^d - m_{\mathcal{F}}$ squares in $\mathcal{S}_1(F_0)$. Given F_1 , F_2 is obtained by removing the same pattern from each of the squares in $\mathcal{S}_1(F_1)$. Iterating, we obtain a sequence (F_n) , where F_n is the union of $m_{\mathcal{F}}^n$ squares in $\mathcal{S}_n(F_0)$. Formally, we define

$$F_{n+1} = \bigcup_{S \in \mathcal{S}_n(F_n)} \Psi_S(F_1) = \bigcup_{S \in \mathcal{S}_1(F_1)} \Psi_S(F_n), \quad n \geq 1.$$

We call the set $\mathcal{F} = \bigcap_{n=0}^{\infty} F_n$ a *Sierpinski carpet*.

Set

$$\mathcal{P} = \bigcup_{r=0}^{\infty} l_{\mathcal{F}}^r F_r.$$

We call \mathcal{P} the *pre-carpet* (see [O]). We define the *unbounded scaled pre-carpet* \mathcal{P}_N by

$$\mathcal{P}_N = l_{\mathcal{F}}^{-N} \mathcal{P} = \bigcup_{r=0}^{\infty} l_{\mathcal{F}}^{r-N} F_r, \quad N \geq 0.$$

Until the end of Section 4 we will fix $N \geq 0$, and work on the scaled pre-carpet \mathcal{P}_N . Any dependence of constants on N will be given explicitly.

We will require a certain amount of notation to describe various subsets of \mathcal{P}_N . Let \mathcal{S}_n^* be the set of cubes in \mathbb{R}^d of side length $2l_{\mathcal{F}}^{-n}$ which are unions of 2^d cubes in \mathcal{S}_n . For $x \in \mathcal{P}_N$ let $Q(x)$ be the cube in \mathcal{S}_n^* with center closest to x . (We use some procedure to break ties.) Set $D_n(x) = Q(x) \cap \mathcal{P}_N$.

If $x = (x_1, \dots, x_d)$ is a point in \mathbb{R}^d , write $\|x\|_{l^\infty} = \max_{1 \leq i \leq d} |x_i|$. For $x, y \in \mathcal{P}_N$ let $d(x, y)$ denote the length of the shortest path (i.e., geodesic) in \mathcal{P}_N connecting x and y , where the length of the path is measured in terms of the l^∞ norm. We have

$$d(x, y) = \lim_{\delta \rightarrow 0} \left(\inf \left\{ \sum_{i=0}^m \|x_i - x_{i-1}\|_{l^\infty} : x_0 = x, x_m = y, \|x_i - x_{i-1}\|_{l^\infty} < \delta, x_i \in \mathcal{P}_N \right\} \right).$$

We write $B(x, r) = \{y \in \mathcal{P}_N : d(x, y) < r\}$. Note that the boundary of $B(x, r)$ is a finite union of flat surfaces orthogonal to the axes. We write $\text{diam}(A)$ for the diameter of A in the metric d , and $\text{dist}(A, B)$ for the distance between the sets A and B . Then notation $x \cdot y$ is used for the standard inner product in \mathbb{R}^d .

For $A \subset \mathbb{R}^d$ we write A° , $\text{cl}(A)$, A^c for the usual interior, closure and complement of A , respectively. Let D be a relatively open set in \mathcal{P}_N (in the metric d). We write $\partial_a D$ for the relative boundary of D in \mathcal{P}_N , and $\partial_e D$ for the boundary of D in \mathbb{R}^d . Set $\partial_r D = \partial_e D - \partial_a D$. We use the subscripts a, e, r as mnemonics for ‘absorbing’, ‘everywhere’, and ‘reflecting’, respectively. If $S \in \mathcal{S}_n$ for some $n \in \mathbb{Z}$, and $S^\circ \cap \mathcal{P}_N$ is non-empty we call $S \cap \mathcal{P}_N$ a *special \mathcal{P}_N cube*. Note that if Q is any special \mathcal{P}_N cube of side length $l_{\mathcal{F}}^{-n}$ then Q is isomorphic to $l_{\mathcal{F}}^{-n} F_{N-n}$, where we write $F_{-k} = [0, 1]^d$, $k \geq 1$. If $B = B(x, r)$ then we define $B^* = B(x, 2r)$. Let $I = S \cap \mathcal{P}_N$ be a special \mathcal{P}_N cube. Write S' for the cube with side length 3 times that of S with the same center as S and faces parallel to those of S . Let $I^* = \text{cl}((S' \cap \mathcal{P}_N)^\circ)$.

Let $Q \subset \mathbb{R}^d$ be a cube with edges parallel to the axes. We call any set of the form $Q \cap \mathcal{P}_N$ a \mathcal{P}_N -*cube*. Note that \mathcal{P}_N cubes do not have to be connected.

We define the resistance constant R_n by

$$R_n^{-1} = \inf \left\{ \int_{l_{\mathcal{F}}^n F_n} |\nabla f|^2 dx : f = 0 \text{ on } x_1 = 0, f = 1 \text{ on } x_1 = l_{\mathcal{F}}^n \right\}.$$

Thus R_n is the resistance between two opposite faces of the set $l_{\mathcal{F}}^n F_n$. It is known (see [BB3], [McG]) that there exists a constant $\rho_{\mathcal{F}}$ and constants c_1, c_2 such that

$$c_1 \rho_{\mathcal{F}}^n \leq R_n \leq c_2 \rho_{\mathcal{F}}^n.$$

Let $t_{\mathcal{F}} = (m_{\mathcal{F}})(\rho_{\mathcal{F}})$. We define the fractal dimension, dimension of the walk, and spectral dimension of \mathcal{F} by

$$\begin{aligned} d_f &= \log m_{\mathcal{F}} / \log l_{\mathcal{F}}, \\ d_w &= \log t_{\mathcal{F}} / \log l_{\mathcal{F}}, \\ d_s &= 2d_f / d_w = 2 \log m_{\mathcal{F}} / \log t_{\mathcal{F}}. \end{aligned}$$

d_f is the Hausdorff dimension (and also the packing dimension) of \mathcal{F} . We remark (see [BB3, Remark 5.4]) that we have $d_w > 2$. We will also use

$$\zeta = \frac{l_{\mathcal{F}}^2}{t_{\mathcal{F}}} = l_{\mathcal{F}}^{2-d_w}. \quad (2.1)$$

Since $d_w > 2$ we have $\zeta < 1$. Let

$$\kappa = \frac{m_{\mathcal{F}}}{l_{\mathcal{F}}^d} = l_{\mathcal{F}}^{d_f-d}.$$

This is the Lebesgue measure of F_1 ; we have $\kappa < 1$.

Let $|A|$ denote the Lebesgue measure of a Borel set A . If Q is a special \mathcal{P}_N cube of side length $s = l_{\mathcal{F}}^{-n}$ then it is easy to check that

$$|Q| = \begin{cases} s^d, & \text{if } s \leq l_{\mathcal{F}}^{-N}, \\ \kappa^N s^{d_f}, & \text{if } s \geq l_{\mathcal{F}}^{-N}. \end{cases}$$

As the ball $B(x, r)$ contains a special \mathcal{P}_N cube of side length $s \geq c_1 r$, and can be covered by c_2 or fewer special \mathcal{P}_N cubes of side length $s \in [r, r l_{\mathcal{F}})$ we deduce that

$$|B(x, r)| \asymp \begin{cases} r^d, & \text{if } r \leq l_{\mathcal{F}}^{-N}, \\ \kappa^N r^{d_f}, & \text{if } r \geq l_{\mathcal{F}}^{-N}. \end{cases} \quad (2.2)$$

Note that this implies that Lebesgue measure on \mathcal{P}_N has the volume doubling property:

$$|B(x, 2r)| \leq c_1 |B(x, r)|, \quad r > 0.$$

Let W_t be Brownian motion on \mathcal{P}_N with normal reflection on the boundary of \mathcal{P}_N . Define $Y_t^N = W(\zeta^{-N} t)$. Then Y_t^N is a process on \mathcal{P}_N with generator $\frac{1}{2} \zeta^{-N} \Delta$, and Green function that is ζ^N times that of W_t .

If D is a domain in \mathcal{P}_N (so $D \subset \mathcal{P}_N$, D is connected and relatively open in \mathcal{P}_N) write $u_D(x, y)$ for the Green function of Y^N on D . Then u_D is symmetric, continuous except on the diagonal $\{x = y\}$ and satisfies

$$\frac{1}{2}\Delta u_D(x, y) = -\zeta^N \delta_x(y), \quad x, y \in D$$

in the distributional sense, where δ_x is point mass at x . If D is suitably regular (such as a \mathcal{P}_N -cube or a ball) then we have $u_D(x, y) \rightarrow 0$ as $y \rightarrow \partial_a D$; we extend u_D to $\mathcal{P}_N \times \mathcal{P}_N$ by taking it to be zero off $D \times D$.

Let D be a domain in \mathcal{P}_N , and $A \subset D$. Define

$$U(x, A, D) = \int_A u_D(x, y) dy = \mathbb{E}^x \int_0^{\tau_D} 1_A(Y_s^N) ds;$$

here $\tau_D = \inf\{s > 0 : Y_s^N \in D^c\}$. Note that U is monotone in A and D : if $A \subset A' \subset D \subset D'$ then

$$U(x, A, D) \leq U(x, A', D) \leq U(x, A', D'). \quad (2.3)$$

Define the function

$$\psi(r) = \begin{cases} r^{d_w}, & \text{if } r \geq l_{\mathcal{F}}^{-N}, \\ \zeta^N r^2 & \text{if } r \leq l_{\mathcal{F}}^{-N}. \end{cases}$$

Note that we can also write $\psi(r) = r^{d_w} \vee \zeta^N r^2$.

Lemma 2.2. *Let B be either a special \mathcal{P}_N cube of side length r or a ball $B(x_0, r)$. Then*

- (a) $U(x, B, B) \leq c_1 \psi(r)$ for $x \in \mathcal{P}_N$,
- (b) $U(x, B^*, B^*) \geq c_2 \psi(r)$ for $x \in B$.
- (c) $U(x, B, B^*) \geq c_3 \psi(r)$ for $x \in B$.

(d) If $\frac{1}{2}t \leq s < t$ then

$$U(x, B(x_0, s), B(x_0, t)) \geq c_4 \psi(t - s) \text{ for } x \in B(x_0, s).$$

Proof. (a) and (b) follow from the estimates on hitting times of sets in [BB3], Proposition 5.5.

(c) We just do the case when $B = B(x_0, r)$; the result for special \mathcal{P}_N cubes is very similar. Let $B_1 = B(x_0, r/2)$. Using the Markov property of Y^N and writing $T_1 = \inf\{t \geq 0 : Y_t^N \in B_1\}$, $\tau = \inf\{t \geq 0 : Y_t^N \notin B^*\}$, we have for $x \in B$,

$$\begin{aligned} U(x, B, B^*) &\geq \mathbb{P}^x(T_1 < \tau) \inf_{y \in B_1} U(y, B, B^*) \\ &\geq \mathbb{P}^x(T_1 < \tau) \inf_{y \in B_1} U(y, B(y, r/2), B(y, r/2)) \\ &\geq c_5 \mathbb{P}^x(T_1 < \tau) \psi(r/2). \end{aligned}$$

Here we used (2.3) and (b) to obtain the final line. It follows from the estimates on the transition density of Y^N given in [BB3], Section 6, that there exists $c_6 > 0$ such that $\mathbb{P}^x(T_1 < \tau) > c_6$. Since $\psi(r/2) \geq c_7 \psi(r)$, (c) follows.

(d) Let $y \in B(x_0, s)$. We can find a point z on the geodesic connecting y and x_0 such that $y \in B(z, t - s) \subset B(x_0, s)$. Then $B(z, 2(t - s)) \subset B(x_0, t)$, so, using (2.3),

$$c_8\psi(t - s) \leq U(y, B(z, t - s), B(z, 2(t - s))) \leq U(y, B(x_0, s), B(x_0, t)).$$

□

The following result generalizes Theorem 5.3 of [BB1]. Since there is an error in the proof of that result, we give details of the proof.

Lemma 2.3. *There exist constants $c_1, \beta > 0$ such that if D is a \mathcal{P}_N -cube with side length less than $l_{\mathcal{F}}^2$ and A is a Borel subset of D*

$$|U(x, A, D) - U(y, A, D)| \leq c_1|x - y|^\beta, \quad x, y \in D. \quad (2.4)$$

Proof. Let $f = 1_A$ and write

$$U_D f(x) = \int_D u_D(x, y) f(y) dy = U(x, A, D).$$

For $B \subset \mathcal{P}_N$ let

$$\tau_B = \inf\{t \geq 0 : Y_t^N \in B^c\}$$

be the first exit time of Y^N from B .

By a proof almost identical with that of Theorem 5.2 of [BB1], there exist constants $c_2, \beta_1 > 0$ such that if D is a \mathcal{P}_N -cube with side length less than $l_{\mathcal{F}}^2$, then

$$\mathbb{E}^x \tau_D \leq c_2 d(x, \partial_a D)^{\beta_1}, \quad x \in D. \quad (2.5)$$

Fix $x, y \in D$. If $4d(x, y)^{1/2} > \text{dist}(x, \partial_a D)$, then

$$|U_D f(x) - U_D f(y)| \leq \mathbb{E}^x \tau_D + \mathbb{E}^y \tau_D \leq c_3 d(x, y)^{\beta_1/2}$$

by (2.5), and the theorem is proved in this case..

Now look at the case where $4d(x, y)^{1/2} \leq \text{dist}(x, \partial_a D)$. Let $\delta = 4d(x, y)^{1/2}$ and let $B = B(x, \delta)$. Then $B \subset D$ and if $z \in B$, then $\tau_B \leq \tau_D$. So by the strong Markov property

$$\begin{aligned} U_D f(z) &= \mathbb{E}^z \int_0^{\tau_B} f(Y_t^N) dt + \mathbb{E}^z \int_{\tau_B}^{\tau_D} f(Y_t^N) dt \\ &= \mathbb{E}^z \int_0^{\tau_B} f(Y_t^N) dt + \mathbb{E}^z U_D f(Y_{\tau_B}^N), \quad z \in B. \end{aligned} \quad (2.6)$$

The function $z \rightarrow \mathbb{E}^z U_D f(Y_{\tau_B}^N)$ is harmonic in B . The elliptic Harnack inequality for Y^N (see Theorems 4.2 and 4.3 of [BB3]) implies there exist constants c_4, β_2 such that for $x', y' \in B(x, \delta/2)$

$$|\mathbb{E}^{x'} U_D f(Y_{\tau_B}^N) - \mathbb{E}^{y'} U_D f(Y_{\tau_B}^N)| \leq c_4 \left(\frac{d(x', y')}{\delta} \right)^{\beta_2} \|U_D f\|_{\infty}. \quad (2.7)$$

By (2.5) we have

$$|U_D f(z)| \leq \|f\|_{\infty} \mathbb{E}^z \tau_D \leq c_5, \quad z \in D,$$

and

$$\mathbb{E}^z \int_0^{\tau_B} f(Y_t^N) dt \leq \|f\|_{\infty} \mathbb{E}^z \tau_B \leq c_6 \delta^{\beta_1}. \quad (2.8)$$

Combining (2.6), (2.7) with $x' = x$ and $y' = y$, and (2.8) with z first equal to x and then equal to y , we obtain our result in this case also. \square

There is a Poincaré inequality for \mathcal{P}_N which may be stated in the following form.

Lemma 2.4. *Let B be either a special \mathcal{P}_N cube of side length r or a ball of radius r , and let $I = B$ or $I = B^*$. Suppose the gradient of f is square integrable over I . Then, writing $f_I = |I|^{-1} \int_I f$,*

$$\int_I |f - f_I|^2 \leq c_1 \psi(r) \zeta^{-N} \int_I |\nabla f|^2. \quad (2.9)$$

Proof. If $N = 0$ and $I = D_n(x)$ for some $n \leq 0$ then this is Proposition 7.10 of [BB3]. The case $N = 0$ and $I = D_n(x)$ with $n > 0$ is the usual Poincaré inequality in \mathbb{R}^d . The same argument as in [BB3] also proves this for special \mathcal{P}_N cubes. The case with $N \geq 1$ follows easily by scaling.

To obtain (2.9) when I is a ball we use the argument of Jerison [J], Section 5. We write $s(D_n(x))$ for the diameter (in the metric d) of the set $D_n(x)$: we have $s(D_n(x)) \asymp l_{\mathcal{F}}^{-n}$. Then it is quite straightforward to find a Whitney decomposition $\mathcal{F} = \{D_i = D_{n_i}(x_i), i \geq 1\}$ of I with the following properties:

$$\begin{aligned} & \text{The sets } D_i^o \text{ are pairwise disjoint,} \\ & I = \cup_i D_{n_i-2}(x_i), \\ & \text{For each } y \in I, \quad \#\{i : y \in D_{n_i-4}(x_i)\} \leq c_2, \\ & l_{\mathcal{F}}^6 \leq \text{dist}(x_i, \partial_a I) / s(D_{n_i}(x_i)) \leq l_{\mathcal{F}}^{12}, \quad i \geq 1. \end{aligned}$$

Then, working with the sets D_i rather than balls, the remainder of Jerison's argument follows, with only minor changes, to give (2.9) for balls. \square

3. Sobolev and other inequalities.

We continue to work on \mathcal{P}_N , and will mention explicitly any dependence on N in our estimates.

For the remainder of this section we fix two \mathcal{P}_N -cubes $Q(h) \subset Q(k)$ such that $h = \text{diam}(Q(h)) \leq k = \text{diam}(Q(k)) \leq l_{\mathcal{F}}^2$. Set

$$\begin{aligned} r(x) &= U(x, Q(h), Q(k)), \quad x \in \mathcal{P}_N, \\ \gamma &= 1 + \zeta^{-N} |\nabla r|^2. \end{aligned}$$

Note that $r = 0$ off $Q(k)$, r is strictly positive on $Q(h)$, and $\frac{1}{2}\Delta r = -\zeta^N 1_{Q(h)}$.

Lemma 3.1. (a) r satisfies the bound

$$r(x) \leq c_1, \quad x \in \mathcal{P}_N.$$

(b) There exists c_2 such that

$$\int_{Q(k)} |\nabla r|^2 \leq c_2 \zeta^N |Q(h)| \sup_{Q(k)} |r|.$$

Proof. (a) is immediate from Lemma 2.2, and the fact that $Q(k)$ is contained in a ball of radius $l_{\mathcal{F}}^2$. For (b) note that $r = 0$ on $\partial_a Q(k)$ and $\partial r / \partial n = 0$ on $\partial_r Q(k)$. So by Green's first identity in the domain $Q(k)$,

$$\int_{Q(k)} |\nabla r|^2 = - \int_{Q(k)} r \Delta r = 2\zeta^N \int_{Q(k)} r 1_{Q(h)}.$$

The result is now immediate. □

We will need the following elementary lemma.

Lemma 3.2. Let $x, y, z \geq 0$. If $x \leq c_1(x^{1/2}z^{1/2} + y)$, then

$$x \leq 2c_1y + 4c_1^2z.$$

Proof. If $x < 2c_1y$, we are done. If $x \geq 2c_1y$, then $c_1x^{1/2}z^{1/2} \geq x - c_1y \geq x/2$, so $2c_1z^{1/2} \geq x^{1/2}$. □

We begin by proving a weighted Poincaré inequality.

Proposition 3.3. *Let $Q(h), Q(k), r$ be as in Lemma 3.1. Let I be either a special \mathcal{P}_N cube of side length s or a ball of radius s . Suppose f and its gradient are square integrable over I^* . There exists $c_1 > 0$ such that*

$$\int_I f^2 |\nabla r|^2 \leq c_1 s^{2\beta} \left(\int_{I^*} |\nabla f|^2 + \zeta^N \psi(s)^{-1} \int_{I^*} f^2 \right).$$

Proof. Let $\varphi = U(\cdot, I, I^*)$, and write $\Phi_0 = \inf_I \varphi$, $\Phi_1 = \sup_{I^*} \varphi$. By Lemma 2.2 we have

$$c_2 \psi(s) \leq \Phi_0 \leq \Phi_1 \leq c_3 \psi(s).$$

Write

$$\begin{aligned} A &= \int_{I^*} f^2 \varphi^2 |\nabla r|^2, \\ B &= \int_{I^*} \varphi^2 |\nabla f|^2, \\ C &= \int_{I^*} f^2, \\ D &= \int_{I^*} f^2 |\nabla \varphi|^2, \\ E &= \int_{I^*} |\nabla f|^2. \end{aligned}$$

Then

$$\int_I f^2 |\nabla r|^2 \leq (\inf_I \varphi)^{-2} \int_I f^2 |\nabla r|^2 \varphi^2 \leq \Phi_0^{-2} A.$$

We begin by bounding A . Choose $x_0 \in I$. If $I^* \not\subset Q(k)$, set $\tilde{r} = r$. If $I^* \subset Q(k)$, set $\tilde{r} = r - r(x_0)$. In either case we see that there exists a point in I^* at which \tilde{r} is zero. Also, $\nabla \tilde{r} = \nabla r$, which is 0 off $Q(k)$. Set

$$R = \sup_{I^*} \tilde{r}.$$

By Lemma 2.3 $|\tilde{r}(x) - \tilde{r}(y)| \leq c_4 |x - y|^\beta$ if $x, y \in I^*$, and therefore $R \leq c_5 s^\beta$.

We write

$$\begin{aligned} A &= \int_{I^*} f^2 \varphi^2 |\nabla r|^2 = \int_{I^*} f^2 \varphi^2 |\nabla \tilde{r}|^2 = \int_{Q(k)} f^2 \varphi^2 |\nabla \tilde{r}|^2 \\ &= \frac{1}{2} \int_{Q(k)} f^2 \varphi^2 \Delta(\tilde{r}^2) - \int_{Q(k)} f^2 \varphi^2 \tilde{r} \Delta \tilde{r}, \end{aligned} \tag{3.1}$$

where in the last line we used the identity $|\nabla u|^2 = \frac{1}{2} \Delta u^2 - u \Delta u$.

Now consider the first term on the right hand side of (3.1). If $I^* \not\subset Q(k)$, then $\partial(\tilde{r}^2)/\partial n = 2\tilde{r}(\partial\tilde{r}/\partial n) = 2r(\partial r/\partial n)$, which is 0 on $\partial_a Q(k)$ and $\partial_r Q(k)$. If $I^* \subset Q(k)$, then $\partial(\tilde{r}^2)/\partial n = 2\tilde{r}(\partial\tilde{r}/\partial n) = 2\tilde{r}(\partial r/\partial n)$ is 0 on $\partial_r Q(k)$ and $f^2\varphi^2$ is 0 on $\partial_a Q(k)$. So by Green's first identity,

$$\int_{Q(k)} f^2 \varphi^2 \Delta(\tilde{r}^2) = - \int_{Q(k)} \nabla(f^2 \varphi^2) \nabla(\tilde{r}^2).$$

Thus

$$\begin{aligned} A &= -\frac{1}{2} \int_{Q(k)} \nabla(f^2 \varphi^2) \nabla(\tilde{r}^2) - \int_{Q(k)} f^2 \varphi^2 \tilde{r} \Delta \tilde{r} \\ &\leq \left| \int_{Q(k)} \nabla(f^2 \varphi^2) \nabla(\tilde{r}^2) \right| + c_6 R \zeta^N \int_{I^*} f^2 \varphi^2 \\ &\leq 4 \left| \int_{Q(k)} f^2 \varphi (\nabla \varphi) \tilde{r} \nabla \tilde{r} \right| + 4 \left| \int_{Q(k)} f (\nabla f) (\varphi^2) \tilde{r} \nabla \tilde{r} \right| + c_6 R \zeta^N \Phi_1^2 \int_{I^*} f^2 \\ &\leq 4 \left(\int f^2 \varphi^2 \tilde{r}^2 |\nabla \tilde{r}|^2 \right)^{1/2} \left(\int f^2 |\nabla \varphi|^2 \right)^{1/2} \\ &\quad + 4 \left(\int f^2 \tilde{r}^2 |\nabla \tilde{r}|^2 \varphi^2 \right)^{1/2} \left(\int \varphi^2 |\nabla f|^2 \right)^{1/2} + c_6 R \zeta^N \Phi_1^2 C \\ &\leq c_7 (R^2 A)^{1/2} (D^{1/2} + B^{1/2}) + c_6 R \zeta^N \Phi_1^2 C. \end{aligned}$$

As $D^{1/2} + B^{1/2} \leq 2(B + D)^{1/2}$, by Lemma 3.2

$$A \leq c_8 R^2 (B + D) + c_8 \zeta^N R \Phi_1^2 C.$$

We now bound D . We have

$$D = \int_{I^*} f^2 |\nabla \varphi|^2 = \frac{1}{2} \int_{I^*} f^2 \Delta(\varphi^2) - \int_{I^*} f^2 \varphi \Delta \varphi.$$

Since $\partial(\varphi^2)/\partial n = 2\varphi(\partial\varphi/\partial n)$ is 0 on $\partial_r I^*$ and 0 on $\partial_a I^*$, by Green's first identity

$$\int_{I^*} f^2 \Delta(\varphi^2) = - \int_{I^*} \nabla(f^2) \nabla(\varphi^2).$$

Since $|\Delta \varphi|$ is bounded by $2\zeta^N$ on I^* , then $|\int_{I^*} f^2 \varphi \Delta \varphi| \leq c_9 \zeta^N \Phi_1 C$. So

$$\begin{aligned} D &\leq \left| \int_{I^*} \nabla(f^2) \nabla(\varphi^2) \right| + c_9 \zeta^N \Phi_1 C \\ &= 4 \left| \int_{I^*} f (\nabla f) \varphi \nabla \varphi \right| + c_9 \zeta^N \Phi_1 C \\ &\leq 4 \left(\int f^2 |\nabla \varphi|^2 \right)^{1/2} \left(\int \varphi^2 |\nabla f|^2 \right)^{1/2} + c_9 \zeta^N \Phi_1 C \\ &\leq c_{10} (D^{1/2} B^{1/2} + \zeta^N \Phi_1 C). \end{aligned}$$

Using Lemma 3.2 again we conclude that

$$D \leq c_{11}(B + \zeta^N \Phi_1 C).$$

Finally, as $B \leq \Phi_1^2 E$, we deduce that

$$A \leq c_{12} R^2 \Phi_1^2 E + c_{12} \zeta^N (R \Phi_1^2 + R^2 \Phi_1) C.$$

Since $\psi(s) \leq c_{13} s^\beta$, and $R \leq c_{14} s^\beta$ we have,

$$\int_I f^2 |\nabla r|^2 \leq \Phi_0^{-2} A \leq c_{15} (\Phi_1 / \Phi_0)^2 s^{2\beta} E + c_{15} \zeta^N (\Phi_1 / \Phi_0)^2 s^{2\beta} \Phi_1^{-1} C.$$

Using the bounds above on Φ_i the conclusion follows. \square

Corollary 3.4. *Let f , I , and I^* be as in Proposition 3.3.*

(a) *Then if $f_{I^*} = |I^*|^{-1} \int_{I^*} f$,*

$$\int_I (f - f_{I^*})^2 \gamma \leq c_1 \zeta^{-N} s^{2\beta} \int_{I^*} |\nabla f|^2. \quad (3.2)$$

(b) *Further,*

$$\int_I f^2 \gamma \leq c_2 \zeta^{-N} s^{2\beta} \int_{I^*} |\nabla f|^2 + |I|^{-1} \left(\int_I |f| \gamma \right)^2.$$

Proof. Applying Proposition 3.3 to $f - f_{I^*}$ we deduce

$$\int_I (f - f_{I^*})^2 |\nabla r|^2 \leq c_3 s^{2\beta} \left(\int_{I^*} |\nabla f|^2 + \zeta^N \psi(s)^{-1} \int_{I^*} (f - f_{I^*})^2 \right). \quad (3.3)$$

By the Poincaré inequality Lemma 2.4 we have

$$\int_I (f - f_{I^*})^2 \leq \int_{I^*} (f - f_{I^*})^2 \leq c_4 \zeta^{-N} \psi(s) \int_{I^*} |\nabla f|^2. \quad (3.4)$$

Substituting the second inequality of (3.4) into (3.3),

$$\int_I (f - f_{I^*})^2 |\nabla r|^2 \leq c_5 s^{2\beta} \int_{I^*} |\nabla f|^2. \quad (3.5)$$

Since $\psi(s) \leq c_6 s^\beta$ we obtain (a) by adding (3.4) and (3.5).

(b) Now let $b = \int_I f \gamma / \int_I \gamma$. Then

$$\begin{aligned} \int_I f^2 \gamma &= \int_I (f - b)^2 \gamma + b^2 \int_I \gamma \\ &= \int_I (f - b)^2 \gamma + \left(\int_I \gamma \right)^{-1} \left(\int_I f \gamma \right)^2 \\ &\leq \int_I (f - f_{I^*})^2 \gamma + |I|^{-1} \left(\int_I f \gamma \right)^2. \end{aligned} \quad (3.6)$$

Combining (3.2) and (3.6) completes the proof. \square

We can obtain a sharper result if we just consider special \mathcal{P}_N cubes.

Corollary 3.5. *Let I be a special \mathcal{P}_N cube. Then*

$$\int_I f^2 |\nabla r|^2 \leq c_1 s^{2\beta} \left(\int_I |\nabla f|^2 + \zeta^N \psi(s)^{-1} \int_I f^2 \right), \quad (3.7)$$

$$\int_I f^2 \gamma \leq c_1 \zeta^{-N} s^{2\beta} \int_I |\nabla f|^2 + |I|^{-1} \left(\int_I |f| \gamma \right)^2. \quad (3.8)$$

Proof. Note that the left-hand sides of (3.7) and (3.8) do not depend on the values of f outside I . Recall that I^* is the union of the (3^d or fewer) special \mathcal{P}_N -cubes of side length s touching I ; extend f to a function \tilde{f} on I^* by reflection. Then

$$\int_{I^*} \tilde{f}^2 \leq 3^d \int_I f^2, \quad \int_{I^*} |\nabla \tilde{f}|^2 \leq 3^d \int_I |\nabla f|^2,$$

and (3.7) and (3.8) now follow from Proposition 3.3 and Corollary 3.4(b) for \tilde{f} . \square

Next we proceed to a Nash inequality for special \mathcal{P}_N cubes. Because the Laplacian is not a symmetric operator with respect to γ , we cannot use the method in [SC].

Proposition 3.6. *Let J be a special \mathcal{P}_N cube with side length $s \leq l_{\mathcal{F}}^2$. Suppose the gradient of f is square integrable over J and $\int_J f^2 \gamma < \infty$. Then*

$$\int_J f^2 \gamma \leq c_1 \max \left(A^{d_f/(2\beta+d_f)} B^{2\beta/(2\beta+d_f)}, A^{d/(2\beta+d)} (\kappa^N B)^{2\beta/(2\beta+d)} \right)$$

where

$$A = \zeta^{-N} \int_J |\nabla f|^2 + s^{-2\beta} \int_J f^2 \gamma, \quad B = \kappa^{-N} \left(\int_J |f| \gamma \right)^2.$$

Proof. The result is trivial if $A = 0$, so we may assume $A > 0$. Let $t \in (0, s)$. We can find a covering of J by special \mathcal{P}_N cubes I_i of side length between $t/l_{\mathcal{F}}$ and t such that $J = \cup I_i$, and the I_i^c are disjoint. Note that $|I_i| \asymp \kappa^N t^{d_f} \wedge t^d$. Set $\psi_0(t) = t^{-d_f} \vee (\kappa^N t^{-d})$. We apply Corollaries 3.4 and 3.5 and sum. So

$$\begin{aligned} \int_J f^2 \gamma &= \sum_i \int_{I_i} f^2 \gamma \\ &\leq c_2 t^{2\beta} \zeta^{-N} \sum_i \int_{I_i} |\nabla f|^2 + c_2 \sum_i |I_i|^{-1} \left(\int_{I_i} |f| \gamma \right)^2 \\ &\leq c_3 t^{2\beta} \zeta^{-N} \int_J |\nabla f|^2 + c_3 \psi_0(t) \kappa^{-N} \left(\sum_i \int_{I_i} |f| \gamma \right)^2 \\ &\leq c_4 t^{2\beta} A + c_5 \psi_0(t) B. \end{aligned} \quad (3.9)$$

If $t > s$ then (possibly adjusting the constant c_4), the inequality (3.9) is trivial. If we now choose t_0 so that $t_0^{2\beta} A = \psi_0(t_0) B$, then we have that

$$t_0 = \begin{cases} (B/A)^{1/(2\beta+d_f)}, & \text{if } t_0 \geq l_{\mathcal{F}}^{-N}, \\ (\kappa^N B/A)^{1/(2\beta+d)}, & \text{if } t_0 \leq l_{\mathcal{F}}^{-N}. \end{cases}$$

Now let $t = t_0$ and substitute in (3.9) to conclude the proof. \square

Next is a preliminary version of a weighted Sobolev inequality. Again the lack of symmetry of the Laplacian with respect to γ necessitates new methods.

Proposition 3.7. *Let J be a special \mathcal{P}_N cube with side length $s \leq 1$. Let f be as above. Then for any $R \in (2, 2 + 2\beta/d)$ there exists $c_1(R) < \infty$ such that*

$$\left(\kappa^{-N} \int_J |f|^R \gamma \right)^{1/R} \leq c_1(R) \left[\zeta^{-N} \kappa^{-N} \int_J |\nabla f|^2 + \kappa^{-N} s^{-2\beta} \int_J f^2 \gamma \right]^{1/2}. \quad (3.10)$$

Proof. Since $|\nabla(f^+)| \leq |\nabla f|$ a.e. and $|f| \leq f^+ + f^-$, it suffices to consider nonnegative f . Write

$$A_0(f) = \zeta^{-N} \kappa^{-N} \int_J |\nabla f|^2 + \kappa^{-N} s^{-2\beta} \int_J f^2 \gamma, \quad B_0(f) = \left(\kappa^{-N} \int_J |f| \gamma \right)^2.$$

Multiplying f by $A_0(f)^{-1/2}$, it is enough to prove

$$\kappa^{-N} \int_J |f|^R \gamma \leq c_1 \quad \text{if } A_0(f) = 1. \quad (3.11)$$

Set

$$p_n = \kappa^{-N} \int_{\{f \geq 2^n\} \cap J} \gamma.$$

Then

$$p_n \leq p_0 \leq \kappa^{-N} \int_{\{f \geq 1\} \cap J} f^2 \gamma \leq \kappa^{-N} \int_J f^2 \gamma \leq s^{2\beta} A_0(f) \leq 1.$$

Let $f_n = (f \wedge 2^{n+1}) - (f \wedge 2^n)$; note that $f_n \leq 2^n$, that $f_n = 2^n$ on $J \cap \{f \geq 2^{n+1}\}$, and that $f_n = 0$ on $\{f < 2^n\}$. Therefore

$$\kappa^{-N} \int_J f_n \gamma = \kappa^{-N} \int_{\{f \geq 2^n\} \cap J} f_n \gamma \leq \kappa^{-N} \int_{\{f \geq 2^n\} \cap J} \gamma 2^n = 2^n p_n, \quad (3.12)$$

while

$$\kappa^{-N} \int_J f_n^2 \gamma \geq \kappa^{-N} \int_{\{f \geq 2^n\} \cap J} f_n^2 \gamma \geq \kappa^{-N} \int_{\{f \geq 2^{n+1}\} \cap J} f_n^2 \gamma = 2^{2n} p_{n+1}. \quad (3.13)$$

Since $\int_J f_n^2 \gamma \leq \int_J f^2 \gamma$ and $\int_J |\nabla f_n|^2 \leq \int_J |\nabla f|^2$, we have $A_0(f_n) \leq A_0(f)$. So, from (3.13) we deduce

$$\begin{aligned} p_n &\leq 2^{-2(n-1)} \kappa^{-N} s^{-2\beta} \int_J f_n^2 \gamma \\ &\leq 4 \cdot 2^{-2n} A_0(f_n) \leq 4 \cdot 2^{-2n}. \end{aligned}$$

Applying Proposition 3.6 to f_n we have, using the fact that $A_0(f) \leq 1$,

$$\begin{aligned} \int_J f_n^2 \gamma &\leq c_2 \max \left((\kappa^{-N} (\int_J f_n \gamma)^2)^{2\beta/(2\beta+d_f)}, (\int_J f_n \gamma)^{2\beta/(2\beta+d)} \right) \\ &= c_2 \max \left((\kappa^N)^{2\beta/(2\beta+d_f)} B_0(f_n)^{\beta/(2\beta+d_f)}, (\kappa^N)^{2\beta/(2\beta+d)} B_0(f_n)^{\beta/(2\beta+d)} \right), \end{aligned}$$

and so

$$\kappa^{-N} \int_J f_n^2 \gamma \leq c_2 \max \left(B_0(f_n)^{\beta/(2\beta+d_f)}, B_0(f_n)^{\beta/(2\beta+d)} \right).$$

Using (3.12) and (3.13) we obtain

$$2^{2n} p_{n+1} \leq c_3 \max \left((2^n p_n)^{2\beta/(2\beta+d_f)}, (2^n p_n)^{2\beta/(2\beta+d)} \right). \quad (3.14)$$

Since $2^n p_n \leq 4$ and $d_f < d$, both the terms on the right hand side of (3.14) are dominated by $c_4 (2^n p_n)^{2\beta/(2\beta+d)}$. Therefore

$$p_{n+1} \leq c_4 (2^{-n})^{(2\beta+2d)/(2\beta+d)} (p_n)^{2\beta/(2\beta+d)}.$$

Elementary calculations now verify that $p_n \leq a 2^{-n\theta}$ where $\theta = 2(\beta + d)/d$ and $a = c_5 \geq 1$, is a constant depending only on c_4 , β and d .

Since

$$\kappa^{-N} \int_J |f|^R \gamma \leq c_6 \sum_{n=0}^{\infty} 2^{nR} p_n,$$

we deduce (3.11) (with a constant depending on R) for any $R \in (2, 2 + 2\beta/d)$. \square

We can modify slightly the final term in (3.10).

Corollary 3.8. *Let J be a special \mathcal{P}_N cube of side $s \leq l_{\mathcal{F}}^2$. Let f and its gradient be square integrable over J . Then for any $R \in (2, 2 + 2\beta/d)$ there exists $c_1(R) < \infty$ such that*

$$\left(\kappa^{-N} \int_J |f|^R \gamma \right)^{1/R} \leq c_1(R) \left[\zeta^{-N} \kappa^{-N} \int_J |\nabla f|^2 + \kappa^{-N} s^{-d_w} \int_J f^2 \right]^{1/2}. \quad (3.15)$$

Proof. We have, using Corollary 3.5, and the fact that $2\beta \leq 2 < d_w$,

$$\begin{aligned} \int_J f^2 \gamma &= \int_J f^2 + \zeta^{-N} \int_J f^2 |\nabla r|^2 \\ &\leq \int_J f^2 + c_2 \zeta^{-N} s^{2\beta} \int_J |\nabla f|^2 + c_3 s^{2\beta-d_w} \int_J f^2 \\ &\leq c_4 \zeta^{-N} s^{2\beta} \int_J |\nabla f|^2 + c_5 s^{2\beta-d_w} \int_J f^2. \end{aligned}$$

The result now follows from substituting this in the last term of (3.10). \square

We now fix $R \in (2, 2 + 2\beta/d)$.

Theorem 3.9. *Let $Q \subset \mathcal{P}_N$ be the union of a finite number of disjoint special \mathcal{P}_N cubes each of side length $s \leq l_{\mathcal{F}}^2$. Let f and its gradient be square integrable over Q . Then*

$$\left(\kappa^{-N} \int_Q |f|^{R\gamma} \right)^{1/R} \leq c_1(R) \left[\zeta^{-N} \kappa^{-N} \int_Q |\nabla f|^2 + \kappa^{-N} s^{-d_w} \int_Q f^2 \right]^{1/2}. \quad (3.16)$$

Proof. If J_i are the cubes with $\cup J_i = Q$ then, applying Corollary 3.8 to each of the J_i ,

$$\begin{aligned} \kappa^{-N} \int_Q |f|^{R\gamma} &= \sum_{i=1}^M \kappa^{-N} \int_{J_i} |f|^{R\gamma} \\ &\leq c_2 \sum_{i=1}^M \left(\zeta^{-N} \kappa^{-N} \int_{J_i} |\nabla f|^2 + s^{-d_w} \kappa^{-N} \int_{J_i} f^2 \right)^{R/2}. \end{aligned}$$

If $p > 1$ and $x_i > 0$ then $\sum x_i^p \leq (\sum x_i)^p$. So

$$\begin{aligned} \kappa^{-N} \int_Q |f|^{R\gamma} &\leq \left(\sum_{i=1}^M \left[\zeta^{-N} \kappa^{-N} \int_{J_i} |\nabla f|^2 + s^{-d_w} \kappa^{-N} \int_{J_i} f^2 \right] \right)^{R/2} \\ &= \left(\zeta^{-N} \kappa^{-N} \int_Q |\nabla f|^2 + s^{-d_w} \kappa^{-N} \int_Q f^2 \right)^{R/2}. \end{aligned}$$

\square

4. Harnack inequality.

In this section we prove Theorem 1.1. We look at the operator $\mathcal{L} = \sum_{i,j=1}^d \partial_i (a_{ij}(x) \partial_j)$, where the a_{ij} are bounded, strictly elliptic, and smooth. On the boundaries of \mathcal{P}_N we impose conormal reflection. We will show the Harnack inequality for nonnegative \mathcal{L} -harmonic functions with bounds that do not depend on the smoothness of the a_{ij} .

The following result is proved exactly as in Moser [M1], Lemma 4.

Proposition 4.1. *Let $D \subset \mathcal{P}_N$ be a domain in \mathcal{P}_N , and suppose u is \mathcal{L} -harmonic in D , $v = u^k$, where $k \in \mathbb{R}$, $k \neq 1/2$, and η is supported in D° . Suppose the gradient of η is square integrable over D . Then*

$$\int_D \eta^2 |\nabla v|^2 dx \leq c_1 \left(\frac{2k}{2k-1} \right)^2 \int_D |\nabla \eta|^2 v^2 dx.$$

Now let u be \mathcal{L} -harmonic and non-negative in $\mathcal{P}_N \cap [0, l_{\mathcal{F}}]^d$; we assume $u \not\equiv 0$. The usual Harnack inequality in \mathbb{R}^d , combined with the connectedness hypothesis (H2) implies that u and u^{-1} are continuous and bounded (by a constant $c(u, N, \varepsilon)$ depending on u , N and ε) on $\mathcal{P}_N \cap [0, l_{\mathcal{F}} - \varepsilon]^d$ for every $\varepsilon > 0$. By Proposition 4.1, the gradient of powers of u will be square integrable over bounded subsets of \mathcal{P}_N .

Let $y_0 = 1$ and $y_k = 1 - \sum_{i=1}^k l_{\mathcal{F}}^{-i}$ for $1 \leq k \leq \infty$, and set

$$Q_k = [0, y_k]^d \cap \mathcal{P}_N, \quad 0 \leq k \leq \infty.$$

Since $l_{\mathcal{F}} \geq 3$, then $y_\infty \geq \frac{1}{2}$. Note that each Q_k is a \mathcal{P}_N -cube and is the union of at most $l_{\mathcal{F}}^{kd_f}$ special \mathcal{P}_N cubes each of side length $l_{\mathcal{F}}^{-k}$. Note also that $\text{dist}(Q_{k+1}, \mathcal{P}_N - Q_k) = l_{\mathcal{F}}^{-(k+1)}$.

Proposition 4.2. *Let v be either u or u^{-1} . There exists c_1 such that if $0 < q < 2$, then*

$$\sup_{Q_\infty} v^{2q} \leq c_1 \kappa^{-N} \int_{Q_0} (\zeta^{-N} |\nabla v^q|^2 + v^{2q}).$$

Proof. For $0 \leq k < \infty$ let

$$r_k = U(\cdot, Q_{k+1}, Q_k), \quad \gamma_k = 1 + \zeta^{-N} |\nabla r_k|^2.$$

Let $f = u^p$, where $p \in \mathbb{R}$, $p \neq \frac{1}{2}$. Applying Theorem 3.9 we have, writing $S = R/2$,

$$\left(\kappa^{-N} \int_{Q_{k+1}} f^{2S} \gamma_{k+1} \right)^{1/S} \leq c_2 \left[\kappa^{-N} \int_{Q_{k+1}} \zeta^{-N} |\nabla f|^2 + l_{\mathcal{F}}^{(k+1)d_w} \kappa^{-N} \int_{Q_{k+1}} f^2 \right]. \quad (4.1)$$

We start with the first term on the right-hand side of (4.1). If $x \in Q_{k+1}$ then there exists a special \mathcal{P}_N cube I of side $l_{\mathcal{F}}^{-k-1}$ such that $x \in I \subset Q_{k+1}$. Then $I^* \subset Q_k$, so by Lemma 2.2(c) we have $r_k \geq c_3 l_{\mathcal{F}}^{-kd_w}$ on Q_{k+1} . Hence

$$\begin{aligned} \kappa^{-N} \int_{Q_{k+1}} \zeta^{-N} |\nabla f|^2 &\leq c_4 \kappa^{-N} \zeta^{-N} l_{\mathcal{F}}^{2kd_w} \int_{Q_{k+1}} |\nabla f|^2 r_k^2 \\ &\leq c_4 \kappa^{-N} \zeta^{-N} l_{\mathcal{F}}^{2kd_w} \int_{Q_k} |\nabla f|^2 r_k^2 \\ &\leq c_5 \left(\frac{2p}{(2p-1)} \right)^2 \kappa^{-N} \zeta^{-N} l_{\mathcal{F}}^{2kd_w} \int_{Q_k} f^2 |\nabla r_k|^2 \\ &\leq c_5 \left(\frac{2p}{(2p-1)} \vee 1 \right)^2 \kappa^{-N} l_{\mathcal{F}}^{2kd_w} \int_{Q_k} f^2 \gamma_k. \end{aligned} \quad (4.2)$$

Here we used Proposition 4.1 in the third line. If c_5 is taken large enough, the right hand term in (4.2) also dominates the final term in (4.1). Therefore,

$$\left(\kappa^{-N} \int_{Q_{k+1}} f^{2S} \gamma_{k+1}\right)^{1/S} \leq c_6 \left(\frac{2p}{(2p-1)} \vee 1\right)^2 l_{\mathcal{F}}^{2kd_w} \left(\kappa^{-N} \int_{Q_k} f^2 \gamma_k\right). \quad (4.3)$$

Choose $q' > 0$ such that $\inf_{m \in \mathbb{Z}} |q'S^m - \frac{1}{2}| \geq c_7 > 0$. Suppose first that $q_0 = q'S^{-i}$ for some i . Let $p_n = 2q_0 S^n$ for $n \geq 0$, and write

$$\Psi_k = \left[\kappa^{-N} \int_{Q_k} v^{p_k} \gamma_k\right]^{1/p_k}.$$

Note that $p_{k+1}/2S = p_k/2$. Applying (4.3) to $f = v^{p_{k+1}/(2S)} = v^{p_k/2}$ we have

$$\begin{aligned} \Psi_{k+1}^{p_{k+1}/S} &= \left(\kappa^{-N} \int_{Q_{k+1}} v^{p_{k+1}} \gamma_{k+1}\right)^{1/S} \\ &\leq c_8 l_{\mathcal{F}}^{2kd_w} \left(\kappa^{-N} \int_{Q_k} v^{p_k} \gamma_k\right) = c_8 l_{\mathcal{F}}^{2kd_w} \Psi_k^{p_k}, \end{aligned}$$

or

$$\Psi_{k+1} \leq \left(c_8 l_{\mathcal{F}}^{2kd_w}\right)^{1/p_k} \Psi_k.$$

Hence for every m

$$\log \Psi_m \leq \log \Psi_0 + \sum_{k=1}^m p_k^{-1} \log(c_8 l_{\mathcal{F}}^{2kd_w}). \quad (4.4)$$

As the sum in (4.4) converges, and $\sup_{Q_\infty} v \leq \limsup_{m \rightarrow \infty} \Psi_m$, we have

$$\sup_{Q_\infty} v \leq c_9 \left(\kappa^{-N} \int_{Q_0} v^{2q_0} \gamma_0\right)^{1/(2q_0)}. \quad (4.5)$$

Now let $q \in (0, 2)$. We can take $q_0 = q'S^{-i} < q$. Then by Hölder's inequality, and Lemma 3.1

$$\int_{Q_0} v^{2q_0} \gamma_0 \leq \left(\int_{Q_0} v^{2q} \gamma_0\right)^{q_0/q} \left(\int_{Q_0} \gamma_0\right)^{1-q_0/q} \leq c_{10} \left(\int_{Q_0} v^{2q} \gamma_0\right)^{q_0/q}.$$

Thus

$$\sup_{Q_\infty} v^{2q} \leq c_{11} \int_{Q_0} v^{2q} \gamma_0.$$

By Corollary 3.5 this implies

$$\sup_{Q_\infty} v^{2q} \leq c_{12} \kappa^{-N} \int_{Q_0} (\zeta^{-N} |\nabla v^q|^2 + v^{2q}).$$

□

In the argument above we were tied to the cubes Q_k since we needed to use Theorem 3.9. However, in the remainder of this section it will be more convenient to use the balls $B(x, r)$. An easy covering argument gives us

Corollary 4.3. *Let $u > 0$ be \mathcal{L} -harmonic in $B(x_0, 3)$. There exists c_1 , independent of u and x_0 , such that for $0 < q < 2$ and $v = u$ or $v = u^{-1}$*

$$\sup_{B(x_0, 1)} v^{2q} \leq c_1 \kappa^{-N} \int_{B(x_0, 2)} (\zeta^{-N} |\nabla v^q|^2 + v^{2q}). \quad (4.6)$$

We now follow the ideas of Moser [M2] to link the L^∞ norms of u and u^{-1} . Fix $x_0 \in \mathcal{P}_N$, and write $B(r) = B(x_0, r)$. Let $u > 0$ be \mathcal{L} -harmonic in $B(x_0, 4)$. v is either u or u^{-1} .

Corollary 4.4. *Let $1/2 \leq s < t \leq 4$ and let $r_{st} = U(\cdot, B(s), B(t))$, $\gamma_{st} = 1 + \zeta^{-N} |\nabla r_{st}^2|$. Then if $0 < q < \frac{1}{3}$,*

$$\sup_{B(s)} v^{2q} \leq c_1 (t - s)^{-d_w - d_f} \kappa^{-N} \int_{B(t)} v^{2q} \gamma_{st}.$$

Proof. Let $\theta = \frac{1}{4}(t - s)$, $s' = s + 2\theta$. By Corollary 4.3 and scaling, if $B(x, 3\theta) \subset B(4)$ then

$$\sup_{B(x, \theta)} v^{2q} \leq c_2 \kappa^{-N} \zeta^{-N} \theta^{d_w - d_f} \int_{B(x, 2\theta)} |\nabla v^q|^2 + c_2 \theta^{-d_f} \kappa^{-N} \int_{B(x, 2\theta)} v^{2q}. \quad (4.7)$$

We can cover $B(s)$ by a collection of balls $B(x_i, 2\theta) \subset B(s')$ such that no point in $B(s')$ is contained in more than c_3 of these balls. So by (4.7)

$$\sup_{B(s)} v^{2q} \leq c_4 \kappa^{-N} \zeta^{-N} \theta^{d_w - d_f} \int_{B(s')} |\nabla v^q|^2 + c_4 \theta^{-d_f} \kappa^{-N} \int_{B(s')} v^{2q}. \quad (4.8)$$

By Lemma 2.2(d) $r_{st} \geq c_4 \theta^{d_w}$ on $B(s')$. Since r_{st} is supported on $B(t)^o$, we have by Proposition 4.1,

$$\begin{aligned} \zeta^{-N} \int_{B(s')} |\nabla v^q|^2 &\leq c_5 \zeta^{-N} \theta^{-2d_w} \int_{B(s')} |\nabla v^q|^2 r_{st}^2 \\ &\leq c_5 \zeta^{-N} \theta^{-2d_w} \int_{B(t)} |\nabla v^q|^2 r_{st}^2 \\ &\leq c_6 \zeta^{-N} \theta^{-2d_w} \int_{B(t)} |\nabla r_{st}|^2 v^{2q} \\ &\leq c_7 \theta^{-2d_w} \int_{B(t)} v^{2q} \gamma_{st}. \end{aligned}$$

Combining this with (4.8) and noting that $\theta \leq 1$ completes the proof. \square

Now let $w = \log u$. We will need the following estimate.

Proposition 4.5. *Suppose $h \in [1/2, 2]$ and $w = \log u$. There exists c_1 such that*

$$\zeta^{-N} \int_{B(h)} |\nabla w|^2 \leq c_1 |B(h)|.$$

Proof. Again, this is essentially Moser's proof. Let $\phi = U(\cdot, B(h), B(2h))$, and note that by Lemma 2.2 $\phi \geq c_2$ on $B(h)$. So

$$\int_{B(h)} |\nabla w|^2 \leq c_3 \int \phi^2 |\nabla w|^2.$$

We write

$$\begin{aligned} 0 &= \int \frac{\phi^2}{u} \mathcal{L}u = - \int \nabla(\phi^2/u) \cdot a \nabla u \\ &= - \int \left(2 \frac{\phi}{u} \nabla \phi \cdot a \nabla u - \frac{\phi^2}{u^2} \nabla u \cdot a \nabla u \right) \\ &= -2 \int \phi \nabla \phi \cdot a \nabla w + \int \phi^2 \nabla w \cdot a \nabla w. \end{aligned}$$

So

$$\int \phi^2 |\nabla w|^2 \leq c_4 \left| \int \phi \nabla \phi \cdot a \nabla w \right| \leq c_5 \left(\int |\nabla \phi|^2 \right)^{1/2} \left(\int \phi^2 |\nabla w|^2 \right)^{1/2}.$$

Dividing and squaring,

$$\int \phi^2 |\nabla w|^2 \leq c_6 \int |\nabla \phi|^2,$$

and by Lemma 3.1, $\zeta^{-N} \int |\nabla \phi|^2 \leq c_7 |B(h)|$. □

For $\frac{1}{2} \leq h \leq 4$, let $\alpha(h) = \frac{1}{|B(h)|} \int_{B(h)} w$.

Corollary 4.6. *Let $\frac{1}{2} \leq s < t \leq 1$. Then*

$$\int_{\{|w - \alpha(2)| > A\} \cap B(s)} \gamma_{st} \leq \frac{c_1 \kappa^N}{A^2}.$$

Proof. Note first that

$$\int_{\{|w - \alpha(2)| > A\} \cap B(s)} \gamma_{st} \leq \int_{\{|w - \alpha(2)| > A\} \cap B(1)} \gamma_{st}.$$

By Chebyshev's inequality,

$$\begin{aligned} \int_{\{|w - \alpha(2)| > A\} \cap B(1)} \gamma_{st} &\leq A^{-2} \int_{\{|w - \alpha(2)| > A\} \cap B(1)} |w - \alpha(2)|^2 \gamma_{st} \\ &\leq A^{-2} \int_{B(1)} |w - \alpha(2)|^2 \gamma_{st}. \end{aligned}$$

Now apply Corollary 3.4(a) with $Q(h) = B(s)$, $Q(k) = B(t)$, $I = B(1)$ and $I^* = B(2)$; we have

$$\int_{B(1)} (w - \alpha(2))^2 \gamma_{st} \leq c_2 \zeta^{-N} \int_{B(2)} |\nabla w|^2,$$

and by Proposition 4.5 this is bounded by $c_3 \kappa^N$. \square

Without loss of generality, let us multiply u by a constant so that $\int_{B(2)} \log v = \alpha(2) = 0$. Recall that v is either u or u^{-1} and define

$$\varphi(h) = \sup_{B(h)} \log v.$$

Lemma 4.7. *If $\frac{1}{2} \leq s < t \leq 1$, then*

$$\varphi(s) \leq \frac{3}{4} \varphi(t) + c_1 (t - s)^{-d_f - d_w}. \quad (4.9)$$

Proof. Fix t and write φ for $\varphi(t)$. Let $c_2 > e$ satisfy $c_2 = 6 \log c_2$. If $\varphi(t) \leq c_2$ then as $\varphi(\cdot)$ is increasing

$$\varphi(s) \leq \varphi(t) \leq \frac{3}{4} \varphi(t) + \frac{1}{4} c_2,$$

so that (4.9) holds provided $c_1 \geq c_2/4$.

Now suppose $\varphi > c_2$. From Lemma 3.1(b) we see that $\int_{B(t)} \gamma_{st} \leq c_3 \kappa^N$. By Corollary 4.6 and the fact that $v^p \leq e^{p\varphi}$ on $B(t)$,

$$\begin{aligned} \int_{B(t)} v^{2p} \gamma_{st} &= \int_{B(t) \cap \{\log v \geq \varphi/2\}} v^{2p} \gamma_{st} + \int_{B(t) \cap \{\log v < \varphi/2\}} v^{2p} \gamma_{st} \\ &\leq e^{2p\varphi} \int_{B(t) \cap \{\log v \geq \varphi/2\}} \gamma_{st} + e^{p\varphi} \int_{B(t) \cap \{\log v < \varphi/2\}} \gamma_{st} \\ &\leq \frac{4c_2 c_4 e^{2p\varphi}}{\varphi^2} \kappa^N + e^{p\varphi} \int_{B(t)} \gamma_{st} \\ &\leq c_5 \left(\frac{e^{2p\varphi}}{\varphi^2} + e^{p\varphi} \right) \kappa^N. \end{aligned}$$

Let $p = \frac{2}{\varphi} \log \varphi$, so that $e^{p\varphi} = \varphi^2$. As $\varphi > c_2$ we have $p < (2/c_2) \log c_2 = \frac{1}{3}$. So

$$\int_{B(t)} v^{2p} \gamma_{st} \leq 2c_5 e^{p\varphi} \kappa^N.$$

Therefore by Corollary 4.4,

$$\begin{aligned} \varphi(s) &= \frac{1}{2p} \log [\sup_{B(s)} v^{2p}] \\ &\leq \frac{1}{2p} \log \left[c_6 (t - s)^{-d_f - d_w} \kappa^{-N} \int_{B(t)} v^{2p} \gamma_{st} \right] \\ &\leq \frac{1}{2p} \log \left[c_7 (t - s)^{-d_f - d_w} e^{p\varphi} \right]. \end{aligned}$$

So

$$\varphi(s) \leq \frac{1}{2}\varphi(t) \left[1 + \frac{1}{2} \frac{\log(c_7(t-s)^{-d_f-d_w})}{\log \varphi} \right]. \quad (4.10)$$

Without loss of generality we may take c_7 larger than c_2 . If $\varphi(t) \geq c_7(t-s)^{-d_f-d_w}$, then by (4.10) $\varphi(s) \leq \frac{3}{4}\varphi(t)$, and (4.9) is satisfied. If, on the other hand, $\varphi(t) \leq c_7(t-s)^{-d_f-d_w}$, then since $\varphi(s) \leq \varphi(t)$, we have (4.9) satisfied with $c_1 = c_7$. \square

We can now prove the Harnack inequality.

Theorem 4.8. *There exists c_1 such that if u is nonnegative and \mathcal{L} -harmonic in $B(3)$, then*

$$\frac{\sup_{B(1/2)} u}{\inf_{B(1/2)} u} \leq c_1.$$

Proof. We know that u is continuous and bounded in $B(2)$; we need to show we can bound the ratio of the supremum of u to the infimum of u in $B(1/2)$ by a constant not depending on u . Multiplying u by a constant we can assume $\int_{B(2)} \log u = \alpha(2) = 0$. First let $v = u$.

Choose $t_j = 1 - (1/(j+2))$, so that $t_0 = 1/2$ and $t_i \uparrow 1$. Then by Lemma 4.7, writing $\theta = d_f + d_w$,

$$\begin{aligned} \varphi(t_0) &\leq \frac{3}{4}\varphi(t_1) + c_2(t_1 - t_0)^{-\theta} \\ &\leq \left(\frac{3}{4}\right)^2\varphi(t_2) + c_2(t_1 - t_0)^{-\theta} + \frac{3}{4}c_2(t_2 - t_1)^{-\theta} \\ &\leq \dots \\ &\leq \left(\frac{3}{4}\right)^n\varphi(t_n) + \frac{4}{3}\sum_{i=1}^n \left(\frac{3}{4}\right)^i c_2(t_i - t_{i-1})^{-\theta}, \end{aligned}$$

for any $n \geq 0$. Since $\varphi(t_n) \leq \varphi(1) < \infty$, and

$$\sum_{i=1}^{\infty} \left(\frac{3}{4}\right)^i c_2(t_i - t_{i-1})^{-\theta} = c_3 < \infty,$$

we obtain

$$\sup_{B(1/2)} \log u \leq c_3.$$

Now let $v = u^{-1}$; $\log v = -\log u$ so we still have $\int_{B(2)} \log v = 0$. The same argument as above now implies $\sup_{B(1/2)} \log v \leq c_3$, or

$$\inf_{B(1/2)} \log u \geq -c_3.$$

Combining we deduce

$$e^{-c_3} \leq \inf_{B(1/2)} u \leq \sup_{B(1/2)} u \leq e^{c_3},$$

which is what we wanted to prove. \square

Theorem 1.1 follows from Theorem 4.8 by a scaling and covering argument.

Standard arguments now yield

Corollary 4.9. *Suppose $D \subset E$ are open subsets of \mathcal{P} . There exists c_1 depending only on the ratio of $\text{dist}(\partial_a D, \partial_a E)$ to the diameter of D such that if u is nonnegative and harmonic in E , then*

$$u(x) \leq c_1 u(y), \quad x, y \in D.$$

5. Heat kernel estimates.

In this section we study the fundamental solutions $q(t, x, y)$ of the heat equation in \mathcal{P} :

$$\frac{\partial u}{\partial t}(x, t) = \mathcal{L}u(x, t),$$

where u has conormal reflection on $\partial\mathcal{P}$. In probabilistic terms $q(t, x, y)$ is the transition density of the diffusion X on \mathcal{P} with generator \mathcal{L} . We continue to assume the a_{ij} are smooth, although our bounds will not depend on the smoothness.

Let D be a domain in \mathcal{P} . Let $u_{\mathcal{L}, D}(x, y)$ be the Green function for \mathcal{L} for the process killed on exiting D and let $u_{\Delta, D}$ be the corresponding Green function for reflecting Brownian motion in D killed on hitting $\partial_a D$. Let $C_{\mathcal{L}, D}(A)$ and $C_{\Delta, D}(A)$ be the capacities of a set $A \subset D$ with respect to \mathcal{L} and Δ respectively. We have

$$\begin{aligned} C_{\mathcal{L}, D}(A) &= \inf \left\{ \int_D \nabla f \cdot a \nabla f : f = 0 \text{ on } \partial\mathcal{P} - D, f = 1 \text{ on } A \right\} \\ C_{\Delta, D}(A) &= \inf \left\{ \int_D |\nabla f|^2 : f = 0 \text{ on } \partial\mathcal{P} - D, f = 1 \text{ on } A \right\}. \end{aligned}$$

It is immediate from these definitions that

$$\lambda_1 C_{\Delta, D}(A) \leq C_{\mathcal{L}, D}(A) \leq \lambda_2 C_{\Delta, D}(A), \quad (5.1)$$

where λ_1, λ_2 depend only on the bounds on the matrix a in (1.3).

Let Y be either of the processes W or X . We write u_D, C_D for the Green's functions and capacities for Y , and τ_D for the exit time of Y from D . We define $U_D \mu(x) = \int u_D(x, y) \mu(dy)$.

Lemma 5.1. *Let $x \in \mathcal{P}$, $R > 0$, $D = B(x, R)$, and $B_n = B(x, 2^{-n}R)$ for $n \geq 0$. Then*

$$c_1 \sum_{n=1}^{\infty} |B_n| C_D(B_n)^{-1} \leq \mathbb{E}^x \tau_D \leq c_2 \sum_{n=1}^{\infty} |B_n| C_D(B_n)^{-1}. \quad (5.2)$$

Proof. Note first that $u_D(x, \cdot)$ is continuous and harmonic on $D - \{x\}$, and is zero on $\{y : d(x, y) = R\} = \partial_a D$. So by the maximum principle it follows that $u_D(x, \cdot)$ attains its maximum on $B_{n-1} - B_n$ at a point z_n with $d(x, z_n) = 2^{-n}R$.

Let μ_n be the capacitary measure for \overline{B}_n . Thus $C_D(B_n) = \mu_n(\overline{B}_n)$ and $U_D\mu_n \leq 1$ on D , and equals 1 on B_n . We know that μ_n is concentrated on $\partial\overline{B}_n$. We have

$$1 = U_D\mu_n(x) = \int_{\overline{B}_n} u_D(x, z)\mu_n(dz) \leq u_D(x, z_n)C_D(B_n). \quad (5.3)$$

Now $u_D(z_n, \cdot)$ is harmonic in B_{n+1} , so by the Harnack inequality Corollary 4.9 we have $u_D(z_n, y) \asymp u_D(z_n, x)$ on B_{n+1} . Therefore

$$1 \geq U_D\mu_{n+1}(z_n) = \int_{\overline{B}_n} u_D(y, z_n)\mu_{n+1}(dy) \geq c_3 u_D(z_n, x)C_D(B_{n+1}). \quad (5.4)$$

Now write $A_n = B(z_n, 2^{-(n+1)}R)$. Note that $A_n \subset B_n - B_{n+1}$, so that the sets A_n are disjoint. The estimate (2.2) implies that A_n , B_n and B_{n-1} all have comparable volume. By Corollary 4.9 we have $u_D(x, y) \asymp u_D(x, z_n)$ on A_n .

Using these estimates we have

$$\begin{aligned} \mathbb{E}^x \tau_D &= \int_D u_D(x, y)dy \\ &\geq \sum_{n=1}^{\infty} \int_{A_n} u_D(x, y)dy \\ &\geq c_4 \sum_{n=1}^{\infty} \int_{A_n} u_D(x, z_n)dy \geq c_5 \sum_{n=1}^{\infty} |B_n|C_D(B_n)^{-1}. \end{aligned}$$

Also,

$$\begin{aligned} \mathbb{E}^x \tau_D &= \sum_{n=1}^{\infty} \int_{B_{n-1}-B_n} u_D(x, y)dy \\ &\leq c_6 \sum_{n=1}^{\infty} \int_{B_{n-1}-B_n} u_D(x, z_n)dy \\ &\leq c_6 \sum_{n=1}^{\infty} |B_{n-1}|u_D(x, z_n) \\ &\leq c_7 \sum_{n=1}^{\infty} |B_{n+1}|C_D(B_{n+1})^{-1} \leq c_7 \sum_{m=1}^{\infty} |B_m|C_D(B_m)^{-1}. \end{aligned}$$

□

Let $\tau_D^{\mathcal{L}}$ be the time for the process X_t associated to \mathcal{L} to exit D , and let τ_D^{Δ} be the analogous time for the Brownian motion W_t on \mathcal{P} to exit D . Recall from Section 2 the definition $\psi(r) = r^{d_w} \vee \zeta^N r^2$, and from Lemma 2.2 that if $D = B(x_0, R)$ then

$$\mathbb{E}^x \tau_{D^*}^{\Delta} \leq c_1 \psi(R), \quad x \in D^*, \quad \mathbb{E}^x \tau_{D^*}^{\Delta} \geq c_2 \psi(R), \quad x \in D.$$

We have the same bounds for the exit times of X .

Corollary 5.2. *There exist c_1, c_2 such that*

$$\begin{aligned}\mathbb{E}^x \tau_{D^*}^{\mathcal{L}} &\leq c_1 \psi(R), & x \in D^*, \\ \mathbb{E}^x \tau_{D^*}^{\mathcal{L}} &\geq c_2 \psi(R), & x \in D.\end{aligned}$$

Proof. This is immediate from Lemma 5.1 and the comparison for capacities given by (5.1). \square

Theorem 5.3. *Let $q(t, x, y)$ denote the transition densities for X_t . There exist $c_1, \dots, c_8 \in (0, \infty)$ such that if $x, y \in \mathcal{P}$ and*

(a) *$t \geq \max(1, |x - y|)$, then*

$$\begin{aligned}c_1 t^{-d_s/2} \exp\left(-c_2 \left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right) \\ \leq q(t, x, y) \leq c_3 t^{-d_s/2} \exp\left(-c_4 \left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right);\end{aligned}$$

(b) *if $t \leq 1$, then*

$$c_5 t^{-d/2} \exp(-c_6 |x - y|^2/t) \leq q(t, x, y) \leq c_7 t^{-d/2} \exp(-c_8 |x - y|^2/t).$$

(c) *if $t \geq 1$, $|x - y| > t$, then*

$$c_5 t^{-d_s/2} \exp(-c_6 |x - y|^2/t) \leq q(t, x, y) \leq c_7 t^{-d_s/2} \exp(-c_8 |x - y|^2/t).$$

Proof. The upper bound follows the proof in [BB3], Section 6, exactly, using Corollary 5.2. The lower bound is done as follows:

Just as in [BB3], (6.23), we have $q(t, x, x) \geq c_9 t^{-d_s/2}$ for $t \geq 1$. Let D be a \mathcal{P} -cube of side length t^{d_w} and let $q_D(t, x, y)$ denote the transition densities for X_t killed on exiting D . X is a symmetric process, so there is an eigenvalue expansion for q_D :

$$q_D(t, x, y) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y).$$

As in [BB2], Section 7, we deduce (6.21) and (6.22) of [BB3]. Since $q(t, x, y) \geq q_D(t, x, y)$, we argue as in [BB3], Theorem 6.9, and derive the lower bound. \square

We also have a parabolic Harnack inequality for \mathcal{P} . The statement and proof are the same as in [BB3], Theorem 7.12.

Remark. Using the results of this section we can construct a diffusion process on \mathcal{F} corresponding to \mathcal{L} . As in [BB1], Section 5, we use Corollary 5.2 to obtain a tightness estimate. The Harnack inequality, following [BB1], Section 5, implies that λ -resolvents are Hölder continuous. We first take a subsequence as in [BB1], Section 6 to construct a process corresponding to \mathcal{L} on \mathcal{F} when the a_{ij} are smooth. In the case when the a_{ij} are not smooth, we take a_{ij}^n smooth satisfying the same bound and uniform ellipticity as the a_{ij} and take a limit. It is then straightforward, as in [BB3] Section 6, to derive heat kernel bounds and a parabolic Harnack inequality for this process.

References

- [A] D.G. Aronson, Bounds on the fundamental solution of a parabolic equation. *Bull. Amer. Math. Soc.* **73** (1967) 890-896.
- [AC] P. Auscher and T. Coulhon. Gaussian lower bounds for random walks from elliptic regularity. *Ann. Inst. H. Poincaré Probab. Statist.* **35** (1999), 605–630.
- [BB1] M. T. Barlow and R. F. Bass, The construction of Brownian motion on the Sierpinski carpet. *Ann. I. H. Poincaré* **25** (1989) 225–257.
- [BB2] M.T. Barlow and R.F. Bass, Transition densities for Brownian motion on the Sierpinski carpet. *Probab. Th. Rel. Fields* **91** (1992) 307-330.
- [BB3] M.T. Barlow and R.F. Bass, Brownian motion and harmonic analysis on Sierpinski carpets. *Canad. J. Math.* **51** (1999) 673-744.
- [BB4] M.T. Barlow and R.F. Bass. Random walks on graphical Sierpinski carpets. In: *Random walks and discrete potential theory*, ed. M. Piccardello, W. Woess, *Symposia Mathematica XXXIX* Cambridge Univ. Press, Cambridge, 1999.
- [BCG] M. Barlow, T. Coulhon, A. Grigor'yan. Manifolds and graphs with slow heat kernel decay. Preprint.
- [D] T. Delmotte. Parabolic Harnack inequality and estimates of Markov chains on graphs. *Rev. Math. Iberoamericana* **15** (1999), 181-232.
- [FS] E.B. Fabes and D.W. Stroock, A new proof of Moser's parabolic Harnack inequality via the old ideas of Nash. *Arch. Mech. Rat. Anal.* **96** (1986) 327–338.
- [FOT] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*. de Gruyter, Berlin, 1994.
- [Gr] A.A. Grigor'yan. The heat equation on noncompact Riemannian manifolds. *Math. USSR Sbornik* **72** (1992) 47-77.
- [Hu] J.E. Hutchinson, Fractals and self-similarity. *Indiana J. Math.* **30** (1981) 713-747.
- [J] D. Jerison. The Poincaré inequality for vector fields satisfying Hörmander's condition. *Duke Math. J.* **53** (1986), 503-523.
- [Jo] O.D. Jones. RWSG paper.
- [K] S. Kusuoka: Dirichlet forms on fractals and products of random matrices. *Publ. RIMS Kyoto Univ.*, **25**, 659–680 (1989).
- [Man] B. Mandelbrot, *The Fractal Geometry of Nature*. W.H. Freeman, San Francisco, 1982.
- [McG] I. McGillivray, Some applications of Dirichlet forms in probability theory. Ph.D. dissertation, Cambridge Univ., 1992.
- [M1] J. Moser, On Harnack's inequality for elliptic differential equations. *Comm. Pure Appl. Math.* **14** (1961) 577–591.

- [M2] J. Moser, On a pointwise estimate for parabolic differential equations. *Comm. Pure Appl. Math.* **24** (1971) 727–740.
- [O] H. Osada, Isoperimetric dimension and estimates of heat kernels of pre-Sierpinski carpets. *Probab. Th. Rel. Fields* **86** (1990) 469–490.
- [PW] E. Pardoux and R.J. Williams, Symmetric reflected diffusions. *Ann. I. H. Poincaré* **30** (1994) 13–62.
- [SC] L. Saloff-Coste, A note on Poincaré, Sobolev, and Harnack inequalities. *Inter. Math. Res. Notices* (1992) 27–38.
- [Sie] W. Sierpinski, Sur une courbe cantorienne qui contient une image biunivoque et continue de toute courbe donnée. *C.R. Acad. Sci. Paris* **162** (1916) 629-632.
- [St] K.-T. Sturm. Analysis on local Dirichlet spaces III. The parabolic Harnack inequality. *J. Math. Pures. Appl. (9)* **75** (1996), 273-297.