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LETTER TO THE EDITOR

Resistance and spectral dimension of Sierpinski carpets

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Abstract. We announce results which prove the existence of the spectral dimension d_s for Sierpinski carpets in two dimensions. Our method employs the Einstein relation $\tilde{\zeta} = d_\omega - d_f$. Using this, numerical calculations of the resistance of approximations to the Sierpinski carpet yield an accurate estimate for d_s.

There has been much recent interest in dynamical phenomena on fractals, e.g., vibration, diffusion, field theories, etc [1]. These are found to be governed by the spectral dimension, d_s, of the fractal. d_s is the 'density of states' for the fractal, and is generally defined by

$$N(\omega) \sim \omega^{d_s/2} \qquad \text{as } \omega \to \infty \tag{1}$$

where $N(\omega)$ is the number of eigenvalues less than or equal to ω [2]. Restating (1) in more formal language we have

$$d_s = 2 \lim_{\omega\to\infty} \frac{\ln N(\omega)}{\ln \omega}. \tag{2}$$

It is not obvious that the limit in (2) exists. For finitely ramified fractals renormalisation methods can be applied rigorously and exactly, and both establish the existence of the limit and provide techniques for its evaluation [3]. These methods do not work effectively on infinitely ramified fractals such as the Sierpinski carpets (SCs).

In this letter we: (a) announce results which prove the existence of the limit in (2) for the Sierpinski carpets; (b) establish rigorously for the SCs the Einstein relation connecting d_s with the conductivity exponent $\tilde{\zeta}$ [2, 4]:

$$2d_f/d_s = d_f + \tilde{\zeta} \tag{3}$$

and (c) estimate d_s numerically by calculations of conductivities of approximations to the SC.

Following [5] we define a family of SCs. We start with the unit square $F_0 = [0, 1]^2$ in $\mathbb{R}^2$ and divide it into b^2 equal subsquares. We cut out a central symmetric block of l^2 subsquares and denote the set remaining by F_1. Thus $F_1 = F_0 - (\frac{1}{2} - l/2b, \frac{1}{2} + l/2b)^2$. This procedure is next repeated for the subsquares which remain, and is then iterated indefinitely. We denote by F_n the set remaining at the nth stage: F_n consists of $(b^2 - l^2)^n$

subsquares of side b^{-n}. The (b, l) Sierpinski carpet (which we denote by $\mathrm{SC}(b, l)$) is defined by

$$F=\bigcap_{n=0}^{\infty} F_n .$$

F is a fractal subset of $\mathbb{R}^2$, and has fractal dimension $d_f = \ln(b^2 - l^2)/\ln b$.

Consider a thin plate F_n, with outside corners labelled A, B, C, D. A potential difference is applied between sides AB and CD, while a zero flux boundary condition is applied on sides BC and DA. Let R_n be the resistance of F_n; we normalise the conductivity of the material so that $R_0 = 1$.

Theorem 1.

(a) There exists a constant $c_1 > 1$ such that

$$c_1^{-1} R_n R_m \leqslant R_{n+m} \leqslant c_1 R_n R_m \qquad \text{for all } n, m \geqslant 0. \tag{4}$$

(b) There exists $\rho > 0$ such that

$$c_1^{-1} \rho^n \leqslant R_n \leqslant c_1 \rho^n \qquad \text{for all } n \geqslant 0. \tag{5}$$

We give a sketch of the proof below; full details will appear in [6].

Remarks.

1. The method of proof does not give the value of ρ.
2. We expect that $R_n \sim c\rho^n$ as $n \to \infty$, but do not have a proof for this.

The resistance exponent $\tilde{\zeta}$ may be defined by

$$\tilde{\zeta} = \frac{\log \rho}{\log b} = \lim_{n\to\infty} \frac{\log R_n}{\log b^n}. \tag{6}$$

We relate $\tilde{\zeta}$ and d_s by considering the small time asymptotics of diffusion on F. (A construction of this diffusion process $X(t)$, $t \geqslant 0$, was given in [7].) Set

$$d'_\omega = \frac{\ln[(b^2 - l^2)\rho]}{\ln b} \qquad d'_s = 2d_f/d'_\omega$$
$$\Phi(x, t) = t^{-d'_s/2} \exp[-(x^{d'_\omega} t^{-1})^{1/(d'_\omega - 1)}]. \tag{7}$$

We shall see below that d'_s is the spectral dimension of F.

Theorem 2.

(a) X has a probability transition density $p(t, x, y)$ with respect to Hausdorff measure μ on F, so that for $A \subseteq F$,

$$\mathrm{Prob}(X_t \in A | X_0 = x) = \int_A p(t, x, y)\mu(\mathrm{d}y).$$

(b) $p(t, x, y)$ is continuous on $(0, \infty) \times F \times F$.

(c) There exist constants $c_2, \ldots, c_5$ such that for all $x, y \in F$, $0 < t \leqslant 1$,

$$c_2\Phi(c_3|x-y|, t) \leqslant p(t, x, y) \leqslant c_4\Phi(c_5|x-y|, t). \tag{8}$$

This theorem follows from theorem 1 and the various estimates on X in [7, 8] by the techniques used in [9, 10]. Full details will appear in [4].

Remarks.

1. This form of the transition density agrees with that given for the Sierpinski gasket in [9, 10]. See also the remarks in [1, p 710].

2. The cut-off $t \leq 1$ in (8) arises because we are considering diffusion on a bounded set. If we consider instead diffusion on the unbounded carpet $\tilde{F} = \{x: b^{-k}x \in F$ for some $k \geq 0\}$, then (8) holds for $0 < t < \infty$.

Setting $x = y$ in (8) we deduce that $c_2 \leq t^{d_s'/2} p(t, x, x) \leq c_4$. Hence (see [1, p 706] or [10, p 619]) there exist constants c_6, c_7 such that $c_6 \omega^{d_s'/2} \leq N(\omega) \leq c_7 \omega^{d_s'/2}$, and the existence of the limit in (2) is now apparent. Thus $d_s = d_s'$ and so we have also established the Einstein relation (3) for the SCs.

We now sketch the proof of theorem 1. Let G_n be the network (graph) obtained from F_n by replacing each square of side b^{-n} by a crosswire of four resistors (parallel to the axes), each with resistance $\frac{1}{2}$. Let R_n^G be the resistance of G_n; note that $R_0^G = 1$.

Lemma 3. For all $n, m \geq 0$, we have $R_{n+m} \leq 2 R_n^G R_m$.

Proof. Using standard electrical circuit theory [11], R_m can be obtained as the solution to the variational problem of minimising the energy dissipation of a flow of total flux 1 across F_m:

$$R_m = \inf\left\{ \int_{F_m} |I|^2 \text{: } I \text{ is a flow of flux } 1 \right\}.$$

Let I_m denote the flow on F_m at which the minimum is attained. Define I_m^* by reflection in the diagonal $x_1 = x_2$ (see figure 1).

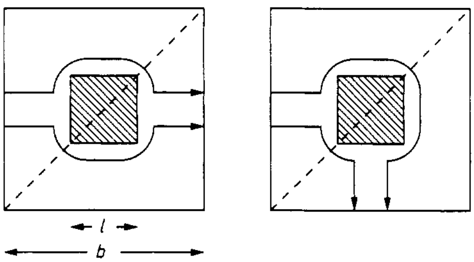

Figure 1. (a) The flow I_m on F_m; (b) the flow I_m^* after reflection across the diagonal. The unit square has been divided into b^2 subsquares, and the central block of l^2 subsquares removed.

By symmetry $\int_{F_m} |I_m^*|^2 = \int_{F_m} |I_m|^2 = R_m$. Now let J_n be the equivalent minimising flow on the network G_n. If we consider one crosswire x in G_n, we will have (signed) flows on the four branches, $K_{x1}, K_{x2}, K_{x3}, K_{x4}$, where $\Sigma_{j=1}^4 K_{xj} = 0$, and the energy dissipation in the crosswire is

$$E_x = \frac{1}{2} \sum_{j=1}^{4} K_{xj}^2 .$$

Write $K_x = \Sigma_{j=1}^4 \max(K_{xj}, 0)$ for the total flux through the crosswire. We now seek to construct a matching flow on the piece of the carpet F_{n+m} corresponding to the crosswire x. So let L_x be a flow on $b^{-n}F_m$ obtained by adding linear combinations of I_m, I_m^*, and their rotations to obtain a flow with inputs $K_{x1}, \ldots, K_{x4}$ on the four outer edges of $b^{-n}F_m$. It is straightforward to check that

$$\int_{b^{-n}F_m} |L_x|^2 = K_x^2 R_m \leq 2R_m E_x. \tag{9}$$

We now construct a flow L on F_{n+m} as follows. Starting with G_n and J_n, replace each crosswire x with a copy of $b^{-n}F_m$, and the flow in the crosswire by the flow L_x. Summing over crosswires we have, from (9),

$$\int_{F_{n+m}} |L|^2 = \sum_x \int_{b^{-n}F_m} |L_x|^2 \leq 2R_m \sum_x E_x = 2R_m R_n^G.$$

Hence $R_{n+m} \leq 2R_m R_n^G$.

The proof of lemma 3 used the characterisation of resistance as the solution to a variational problem. There is a dual characterisation of conductivity as the minimum energy of a potential with total potential difference 1. We shall not go through the details, but the result corresponding to lemma 3 is given by the following.

Lemma 4. $R_{n+m} \geq \frac{1}{243} R_n^G R_m$ for all $n, m \geq 0$.

Proof of theorem 1. (a) is immediate from lemmas 3 and 4, with $c_1 = 486$. Let $x_n = c_1^{-1} R_n$, $y_n = c_1 R_n$; then x_n is supermultiplicative (i.e. $x_{n+m} \geq x_n x_m$) and y_n is submultiplicative. So, using the standard (and easily proved) properties of submultiplicative sequences [12]:

$$\lim_{n\to\infty} \frac{\ln x_n}{n} = \theta = \sup_{n\geq 1} \frac{\ln x_n}{n} \qquad \lim_{n\to\infty} \frac{\ln y_n}{n} = \theta' = \inf_{n\geq 1} \frac{\ln y_n}{n}.$$

As x_n/y_n is constant, $\theta = \theta' = \rho$ say, and (5) is now immediate.

We now turn to the problem of determining the value of ρ. Straightforward shorting and cutting arguments [13] give, for sc(b, l), with $l = bx$

$$\frac{1}{1-x} - x \leq \rho \leq \frac{1}{1-x}. \tag{10}$$

Using (3) this gives bounds on d_s; for sc(3, 1), for example, we deduce

$$1.674 < 2 \ln 8/\ln 12 \leq d_s \leq 2 \ln 8/\ln(28/3) < 1.862. \tag{11}$$

Theorem 1 together with lemmas 3 and 4 implies that both $\ln(R_n)/n$ and $\ln(R_n^G)/n$ converge to $\ln \rho$, so that we can estimate ρ from either the resistance of F_n or G_n. Let H_{nm}, for $n \geq 0$, $m \geq 0$, denote the network obtained from F_n by replacing each subsquare of side b^{-n} by a $b^m \times b^m$ network of crosswires. Thus as $m \to \infty$, H_{nm} converges to the set F_n, and writing $R(H_{nm})$ for the resistance of H_{nm}, we have $R(H_{nn}) = R_n^G$ and $R(H_{n\infty}) = R_n$.

Table 1 gives $R(H_{nm})$ for sc(3, 1) for $1 \leq n \leq 7$, $n \leq m \leq 7$. For $0 \leq n \leq 5$ an estimate of $R(H_{n\infty})$, obtained by Shanks' transform from the last three values, is added.

The computations were performed by Gaussian relaxation over a grid covering one quarter of the carpet. A coarse mesh (with size equal to the smallest holes in the carpet) was used to obtain the diagonal elements of table 1. Successive mesh refinements, in each case by a (linear) factor of 3, were used to improve the approximation to the solution of the differential equation. The largest grid used for sc(3, 1) was

Table 1. $R(H_{nm})$ for SC(3, 1).

n	1	2	3	4	5	6	7
m							
1	1.400 000						
2	1.307 984	1.793 474					
3	1.282 981	1.650 836	2.252 722				
4	1.277 129	1.613 843	2.068 868	2.820 697			
5	1.275 778	1.605 221	2.021 464	2.589 707	3.530 305		
6	1.275 465	1.603 227	2.010 414	2.530 181	3.241 082	4.418 167	
7	1.275 392	1.602 764	2.007 851	2.516 289	3.166 543	4.056 182	5.524 280
∞	1.275 37	1.602 62	2.007 08	2.512 06	3.140 66		

of size 1094×1094. Once a solution had been obtained on this grid, an extra generation of holes was added, and the code reverted to the coarsest possible grid to obtain the next term down the diagonal (table 1). Computation took about 50 h on a SUN 4/110. The computed energy dissipation converged rapidly, but required double precision arithmetic.

One might expect, from intuitive grounds, that R_n/R_{n-1} would converge more rapidly than R_n^G/R_{n-1}^G. However, for a given grid size, one has two more values of the ratio R_n^G/R_{n-1}^G available. Moreover, we have only three entries in the column $n=5$, and the numerical extrapolation appears suspect, which increases further the advantage of computing $a_n = R_n^G/R_{n-1}^G$ rather than R_n/R_{n-1}. For other carpets we therefore merely computed a_n (table 2).

As before, we used Shanks' transform to estimate the value for $n=\infty$. The final row in table 2 gives our estimate for d_s based on these calculations.

In many situations the Einstein relation (3) has been used to obtain $\tilde{\zeta}$ from d_s, where d_s is first estimated by random walk simulations. For the SCs considered here, however, extremely lengthy simulations would be required to achieve results as accurate as those in table 2.

The results of table 1 for the resistances F_1, F_2, F_3 for SC(3, 1) may be compared with the experiments of [14], and we see that the agreement is satisfactory. Note however from tables 1 and 2 that the ratio R_1/R_0 is a poor approximation to ρ. We remark that the values of ρ we obtained are within 1% of $(b^2+l^2)/(b^2-l^2)$, but that equality does not hold.

Table 2. Values of R_n^G/R_{n-1}^G for various SCs.

n	SC(3, 1)	SC(4, 2)	SC(5, 1)	SC(5, 3)
1	1.400 000	1.875 000	1.133 333	2.368 422
2	1.281 053	1.699 436	1.093 240	2.164 505
3	1.256 066	1.675 619	1.089 212	2.142 457
4	1.252 129	1.673 127	1.088 870	2.140 565
5	1.251 572	1.672 879	1.088 841	2.140 394
6	1.251 497			
7	1.251 487			
∞	1.251 49	1.672 85	1.088 84	2.140 38
d_s	1.805 25	1.656 92	1.974 83	1.569 28

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