

Skorokhod Imbedding via Stochastic Integrals

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Given a Brownian motion  $L_t$  and a probability measure  $\mu$  on  $\mathbb{R}$  with mean 0, a Skorokhod imbedding of  $\mu$  is a stopping time  $T$  adapted to the sigma fields of  $L_t$  such that  $L_T$  has distribution  $\mu$ . We give here a new method of constructing such an imbedding using results from the representation of martingales as stochastic integrals.

We first construct a Brownian motion  $N_t$  and a stopping time  $W$  such that  $N_W$  has law  $\mu$ . We then show how, given an arbitrary Brownian motion  $L_t$ , one can construct a stopping time  $T$  such that  $L_T$  has law  $\mu$ .

Define  $p_t(y) = (2\pi t)^{-1/2} e^{-y^2/2t}$ ,  $q_t(y) = \partial p_t(y)/\partial y = -(2\pi t)^{-1/2} (y/t) e^{-y^2/2t}$ .

Let  $X_t$  be a Brownian motion,  $\mathbb{F}_t$  its filtration, and  $g$  a real-valued function.

Lemma 1. Suppose  $E|g(X_1)| < \infty$ . Then

a)  $\sup_{|y| \leq y_0} \int |g(z)| |z-y|^k e^{-(z-y)^2/2t} dz < \infty$  for all positive  $k$ , all  $y_0$ , all  $t < 1$ .

b)  $g(X_1) = Eg(X_1) + \int_0^1 a(s, X_s) dX_s$ , where  $a(s, y) = \int q_{1-s}(z-y) g(z) dz$  for  $s < 1$ ;  
furthermore  $\int_0^1 a^2(s, X_s) ds < \infty$ , a.s.

c)  $E(g(X_1) | \mathbb{F}_s) = b(s, X_s)$  for  $s < 1$ , where  $b(s, y) = \int p_{1-s}(z-y) g(z) dz$ .

Proof. a) follows from the formula for the normal density and the fact that

$$|z-y|^k e^{-(z-y)^2/2t} \leq e^{-z^2/2} \text{ for } z \text{ large.}$$

b) Suppose first that  $g$  is bounded, has compact support, and is in  $C^2$ .

By Clark's formula [1] applied to the functional  $g(X_1)$ ,

$$g(X_1) = Eg(X_1) + \int_0^1 E[g'(X_1) | \mathbb{F}_s] dX_s.$$

(Another derivation of this representation is to use Ito's lemma to take care of the case  $g(x) = e^{iux}$  and then use linearity and a limiting process.)

By the Markov property, if  $s < 1$ ,

$$E[g'(X_1) | \mathcal{F}_s] = \int g'(z) p_{1-s}(X_s - z) dz .$$

An integration by parts gives the result for such  $g$ ; the result for general  $g$  follows by a limit argument.

c) By the Markov property, if  $s < 1$ ,

$$E[g(X_1) | \mathcal{F}_s] = \int g(z) p_{1-s}(X_s - z) dz . \quad \square$$

Lemma 2. Suppose  $g$  is nondecreasing and not identically constant. Then

- a) On compact subsets of  $[0,1) \times \mathbb{R}$ ,  $a(s,y)$  is bounded above, bounded below away from 0, and uniformly Lipschitz in  $s$  and  $y$ .
- b) For each  $s < 1$ ,  $b(s,y)$  is continuous and strictly increasing as a function of  $y$ .
- c) For each  $s < 1$ , let  $B(s,\cdot)$  be the inverse of  $b(s,\cdot)$ ; then on compact subsets of its domain,  $B(s,y)$  is uniformly Lipschitz in  $s$  and jointly continuous in  $s$  and  $y$ .

Proof. a) Suppose  $|y| \leq y_0, s \leq s_0 < 1$ .  $a(s,y)$  is bounded above by lemma 1a. An integration by parts argument shows that  $a(s,y) = \int p_{1-s}(y-z) dg(z)$ , hence  $a$  is bounded below. Using the definition of  $a(s,y)$ , appropriate bounds on  $\partial q_{1-s}/\partial s$  and  $\partial q_{1-s}/\partial y$ , and lemma 1a gives the uniformly Lipschitz result.

b) The definition of  $b$  shows that  $b(s,\cdot)$  is continuous. Since we also have  $b(s,y) = \int g(y+z) p_{1-s}(z) dz$ , it follows that  $b(s,\cdot)$  is nondecreasing, and in fact, strictly increasing since  $g$  is not constant. Note that this implies that the range of  $b(s,\cdot)$  must be an open (possibly infinite) interval.

c) Since  $b(s,\cdot)$  is continuous and strictly increasing, we can define its inverse  $B(s,\cdot)$  on the range of  $b(s,\cdot)$ .  $B(s,y)$  will be continuous in  $y$ .

Integrating by parts,

$$\partial b / \partial y = \int p_{1-s}(y-z) dg(z) ,$$

which is uniformly  $> 0$  for  $s,y$  in a compact subset of  $[0,1) \times \mathbb{R}$ .  $\partial b / \partial s$  is bounded above on compact sets since  $\partial p_{1-s} / \partial s$  is, using lemma 1a again.

We now show that  $B$  is uniformly Lipschitz in  $s$ ,  $s,y$  in a compact subset of the domain of  $B$ . Let  $w = B(s+h,y)$ ,  $x = B(s,y)$ , and suppose  $w \leq x$ ,

the other case being similar. Then

$$0 = b(s+h, w) - b(s, x) = b(s+h, w) - b(s, w) + b(s, w) - b(s, x) \leq C|h| - c(x-w),$$

$$\text{or } |x-w| \leq C|h| / c ,$$

where  $C$  and  $c$  are upper and lower bounds for  $\partial b / \partial s$  and  $\partial b / \partial y$ , respectively.

This proves that  $B$  is uniformly Lipschitz in  $s$ , and it follows immediately that  $B$  is jointly continuous.  $\square$

Now let  $\mu$  be a probability measure on  $\mathbb{R}$  and suppose  $\int |x| d\mu(x) < \infty$  and  $\int x d\mu(x) = 0$ . Let  $F(x) = \mu(-\infty, x]$ , let  $F^{-1}(y) = \inf\{x: F(x) \geq y\}$ , let  $\Phi(x) = \int_{-\infty}^x p_1(y) dy$ , and let  $g(x) = F^{-1}(\Phi(x))$ . Then  $g(X_1)$  has distribution  $\mu$  and  $Eg(X_1) = 0$ .

Define  $M_t = \int_0^t a(s, X_s) dX_s$ , where  $a(s, y)$  is given by lemma 1 for  $s < 1$ ,  $a(s, y) = 1$  for  $s \geq 1$ . Note  $M_1 = g(X_1)$  has law  $\mu$ , and if  $s < 1$ ,  $M_s = b(s, X_s)$ . Let  $R(t) = \int_0^t a^2(s, X_s) ds$ , define  $S(t) = \inf\{r: R(r) \geq t\}$ , and let  $N_t = M_{S(t)}$ . Since the quadratic variation of the continuous martingale  $N$  is  $t$ ,  $N$  is a Brownian motion.

$N_{R(1)} = M_1$ , which has law  $\mu$ . Letting  $W = R(1)$ , it suffices to show that  $R(1)$  is a stopping time of the  $N_t$  process.

**Proposition 3.** (cf. Yershov, [2]).  $(W \geq u)$  is in the right continuous completion of  $\sigma(N_s; s \leq u)$ .

**Proof.** Since  $W = R(1) = \lim_{s \rightarrow 1} R(s)$  by monotone convergence, it suffices to consider  $R(s)$ ,  $s < 1$ .  $(R(s) \geq u) = (s \geq S(u))$ .

It is not hard to see that  $S(t)$  satisfies the equation

$$\frac{dS(t)}{dt} = a^{-2}(S(t), X_{S(t)})$$

if  $S(t) < 1$ . But  $X_{S(t)} = B(S(t), M_{S(t)}) = B(S(t), N_t)$ . Thus, for each  $\omega$ ,  $S(t)$  satisfies the ordinary differential equation

$$(1) \quad \frac{dS(t)}{dt} = a^{-2}(S(t), B(S(t), N_t)) .$$

For each  $\omega$ ,  $\{(S(t), N_t): S(t) \leq s\}$  is contained in a compact subset of the domain of  $B$ . This, lemma 2, and a theorem on uniqueness of solutions of

differential equations [3, pp.1-6] show that there is a unique solution  $S(t)$  to (1) up to the first  $t$  for which  $S(t) = s$ . Moreover, this solution may be constructed via Picard iteration. But then  $(s \geq S(u))$  is in the right continuous completion of  $\sigma(N_s; s \leq u)$  as required.  $\square$

Suppose now that  $L$  is any Brownian motion. We construct an  $L$ -measurable stopping time  $T$  such that  $L$  has law  $\mu$ . Let  $V(t)$  be the unique solution to

$$\frac{dV(t)}{dt} = a^{-2}(V(t), B(V(t), L_t))$$

for each  $\omega$ . (Since  $L_t$  has the same law as  $N_t$ ,  $\{(V(t), L_t): V(t) \leq s\}$  will be in a compact subset of the domain of  $B$ , a.s.) Let  $U(t) = V^{-1}(t)$ ,  $t < 1$  and let  $T = U(1) = \sup_{s < 1} U(s)$ . Clearly the law of  $(L, T)$  is the same as the law of  $(N, W)$ , and so  $L_T$  has distribution  $\mu$ .

$T^{1/2}$  will satisfy certain moment conditions if  $\mu$  does. For example, suppose  $\Psi: [0, \infty) \rightarrow [0, \infty)$  is continuous,  $\int \Psi(|x|) d\mu(x) < \infty$ , and for some  $\epsilon > 0$ ,  $\Psi^{1/(1+\epsilon)}$  is convex and increasing. By Doob's inequality applied to the submartingale  $\Psi^{1/(1+\epsilon)}(|M_s|)$ ,  $E \sup_{s \leq 1} \Psi(|M_s|) < \infty$  since  $E \Psi(|M_1|) < \infty$ . Then by Burkholder's inequality,  $E \Psi(W^{1/2}) < \infty$ .

If  $Y_t$  is a  $d$ -dimensional Brownian motion,  $d \geq 2$ , it is known that there are measures  $\mu$  for which one cannot find a stopping time  $T$  with  $\mu$  the law of  $Y_T$ : just take  $\mu$  atomic and recall that  $d$ -dimensional Brownian motion does not hit points. However, one can always find  $f: \mathbb{R} \rightarrow \mathbb{R}^d$  such that the law of  $f(X_1)$  is  $\mu$ ,  $X_t$  a 1-dimensional Brownian motion. (The coordinate functions of  $f$  are not assumed to be nondecreasing.) One can then use lemma 1 to find a vector-valued function  $A: [0, 1) \times \mathbb{R} \rightarrow \mathbb{R}^d$  such that  $f(X_1) = Ef(X_1) + \int_0^1 A(s, X_s) dX_s$ .

#### REFERENCES

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