A stochastic differential equation with a sticky point

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March 11, 2014

Abstract

Abstract: We consider a degenerate stochastic differential equation that has a sticky point in the Markov process sense. We prove that weak existence and weak uniqueness hold, but that pathwise uniqueness does not hold nor does a strong solution exist.

Subject Classification: Primary 60H10; Secondary 60J60, 60J65

1 Introduction

The one-dimensional stochastic differential equation

$$dX_t = \sigma(X_t)\, dW_t \tag{1.1}$$

has been the subject of intensive study for well over half a century. What can one say about pathwise uniqueness when σ is allowed to be zero at certain points? Of course, a large amount is known, but there are many unanswered questions remaining.

Consider the case where $\sigma(x) = |x|^\alpha$ for $\alpha \in (0, 1)$. When $\alpha \geq 1/2$, it is known there is pathwise uniqueness by the Yamada-Watanabe criterion (see, e.g., [6, Theorem 24.4]) while if $\alpha < 1/2$, it is known there are at least two solutions, the zero solution and one that can be constructed by a non-trivial time change of Brownian motion. However, that is not the end of the story.

In [7], it was shown that there is in fact pathwise uniqueness when $\alpha < 1/2$ provided one restricts attention to the class of solutions that spend zero time at 0.

This can be better understood by using ideas from Markov process theory. The continuous strong Markov processes on the real line that are on natural scale can be characterized by their speed measure. For the example in the preceding paragraph, the speed measure m is given by

$$m(dy) = 1_{(y\neq 0)}|y|^{-2\alpha}\,dy + \gamma\delta_0(dy),$$

where $\gamma \in [0, \infty]$ and δ_0 is point mass at 0. When $\gamma = \infty$, we get the 0 solution, or more precisely, the solution that stays at 0 once it hits 0. If we set $\gamma = 0$, we get the situation considered in [7] where the amount of time spent at 0 has Lebesgue measure zero, and pathwise uniqueness holds among such processes.

In this paper we study an even simpler equation:

$$dX_t = 1_{(X_t\neq 0)}\,dW_t, \qquad X_0 = 0, \tag{1.2}$$

where W is a one-dimensional Brownian motion. One solution is $X_t = W_t$, since Brownian motion spends zero time at 0. Another is the identically 0 solution.

We take $\gamma \in (0, \infty)$ and consider the class of solutions to (1.2) which spend a positive amount of time at 0, with the amount of time parameterized by γ. We give a precise description of what we mean by this in Section 3.

Representing diffusions on the line as the solutions to stochastic differential equations has a long history, going back to Itô in the 1940's, and this paper is a small step in that program. For this reason we characterize our solutions in terms of occupation times determined by a speed measure. Other formulations that are purely in terms of stochastic calculus are possible; see the system (1.5)–(1.6) below.

We start by proving weak existence of solutions to (1.2) for each $\gamma \in (0, \infty)$. We in fact consider a much more general situation. We let m be any measure that gives finite positive mass to each open interval and define the notion of continuous local martingales with speed measure m.

We prove weak uniqueness, or equivalently, uniqueness in law, among continuous local martingales with speed measure m. The fact that we have uniqueness in law not only within the class of strong Markov processes but also within the class of continuous local martingales with a given speed measure may be of independent interest.

We then restrict our attention to (1.2) and look at the class of continuous martingales that solve (1.2) and at the same time have speed measure m, where now

$$m(dy) = 1_{(y \neq 0)}\, dy + \gamma \delta_0(dy) \tag{1.3}$$

with $\gamma \in (0, \infty)$.

Even when we fix γ and restrict attention to solutions to (1.2) that have speed measure m given by (1.3), pathwise uniqueness does not hold. The proof of this fact is the main result of this paper. The reader familiar with excursions will recognize some ideas from that theory in the proof.

Finally, we prove that for each $\gamma \in (0, \infty)$, no strong solution to (1.2) among the class of continuous martingales with speed measure m given by (1.3) exists. Thus, given W, one cannot find a continuous martingale X with speed measure m satisfying (1.2) such that X is adapted to the filtration of W. A consequence of this is that certain natural approximations to the solution of (1.2) do not converge in probability, although they do converge weakly.

Besides increasing the versatility of (1.1), one can easily imagine a practical application of sticky points. Suppose a corporation has a takeover offer at \$10. The stock price is then likely to spend a great deal of time precisely at \$10 but is not constrained to stay at \$10. Thus \$10 would be a sticky point for the solution of the stochastic differential equation that describes the stock price.

Regular continuous strong Markov processes on the line which are on natural scale and have speed measure given by (1.3) are known as sticky Brownian motions. These were first studied by Feller in the 1950's and Itô and McKean in the 1960's.

A posthumously published paper by Chitashvili ([9]) in 1997, based on a technical report produced in 1988, considered processes on the non-negative real line that satisfied the stochastic differential equation

$$dX_t = 1_{(X_t \neq 0)}\, dW_t + \theta 1_{(X_t = 0)}\, dt, \qquad X_t \geq 0, \quad X_0 = x_0, \tag{1.4}$$

with $\theta \in (0, \infty)$. Chitashvii proved weak uniqueness for the pair (X, W) and showed that no strong solution exists.

Warren (see [23] and also [24]) further investigated solutions to (1.4). The process X is not adapted to the filtration generated by W and has some "extra randomness," which Warren characterized.

While this paper was under review, we learned of a preprint by Engelbert and Peskir [11] on the subject of sticky Brownian motions. They considered the system of equations

$$dX_t = 1_{(X_t \neq 0)}\, dW_t, \tag{1.5}$$

$$1_{(X_t = 0)}\, dt = \frac{1}{\mu}\, d\ell_t^0(X), \tag{1.6}$$

where $\mu \in (0, \infty)$ and ℓ_t^0 is the local time in the semimartingale sense at 0 of X. (Local times in the Markov process sense can be different in general.) Engelbert and Peskir proved weak uniqueness of the joint law of (X, W) and proved that no strong solution exists. They also considered a one-sided version of this equation, where $X \geq 0$, and showed that it is equivalent to (1.4). Their results thus provide a new proof of those of Chitashvili.

It is interesting to compare the system (1.5)–(1.6) investigated by [11] with the SDE considered in this paper. Both include the equation (1.5). In this paper, however, in place of (1.6) we use a side condition whose origins come from Markov process theory, namely:

$$\begin{aligned} &X \text{is a continuous martingale with speed measure} \\ &\qquad m(dx) = dx + \gamma\delta_0(dx), \end{aligned} \tag{1.7}$$

where δ_0 is point mass at 0 and "continuous martingale with speed measure m" is defined in (3.1). One can show that a solution to the system studied by [11] is a solution to the formulation considered in this paper and vice versa, and we sketch the argument in Remark 5.3. However, we did not see a way of proving this without first proving the uniqueness results of this paper and using the uniqueness results of [11].

Other papers that show no strong solution exists for stochastic differential equations that are closely related include [1], [2], and [15].

After a short section of preliminaries, Section 2, we define speed measures for local martingales in Section 3 and consider the existence of such local martingales. Section 4 proves weak uniqueness, while in Section 5 we prove that continuous martingales with speed measure m given by (1.3) satisfy (1.2). Sections 6, 7, and 8 prove that pathwise uniqueness and strong existence fail. The first of these sections considers some approximations to a solution to (1.2), the second proves some needed estimates, and the proof is completed in the third.

Acknowledgment. We would like to thank Prof. H. Farnsworth for suggesting a mathematical finance interpretation of a sticky point.

2 Preliminaries

For information on martingales and stochastic calculus, see [6], [14] or [22]. For background on continuous Markov processes on the line, see the above references and also [5], [13], or [16].

We start with an easy lemma concerning continuous local martingales.

Lemma 2.1. *Suppose X is a continuous local martingale which exits a finite non-empty interval I a.s. If the endpoints of the interval are a and b, $a < x < b$, and $X_0 = x$ a.s., then*

$$\mathbb{E}\, \langle X \rangle_{\tau_I} = (x-a)(b-x),$$

where τ_I is the first exit time of I and $\langle X \rangle_t$ is the quadratic variation process of X.

Proof. Any such local martingale is a time change of a Brownian motion, at least up until the time of exiting the interval I. The result follows by performing a change of variables in the corresponding result for Brownian motion; see, e.g., [6, Proposition 3.16]. □

Let I be a finite non-empty interval with endpoints $a < b$. Each of the endpoints may be in I or in I^c. Define $g_I(x, y)$ by

$$g_I(x,y) = \begin{cases} 2(x-a)(b-y)/(b-a), & a \le x < y \le b; \\ 2(y-a)(b-x)/(b-a), & a \le y \le x \le b. \end{cases}$$

Let m be a measure such that m gives finite strictly positive measure to every finite open interval. Let

$$G_I(x) = \int_I g_I(x,y)\, m(dy).$$

If X is a real-valued process adapted to a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions, we let

$$\tau_I = \inf\{t > 0 : X_t \notin I\}. \tag{2.1}$$

When we want to have exit times for more than one process at once, we write $\tau_I(X)$, $\tau_I(Y)$, etc. Define

$$T_x = \inf\{t > 0 : X_t = x\}. \tag{2.2}$$

A continuous strong Markov process $(X, \mathbb{P}^x)$ on the real line is regular if $\mathbb{P}^x(T_y < \infty) > 0$ for each x and y. Thus, starting at x, there is positive probability of hitting y for each x and y. A regular continuous strong Markov process X is on natural scale if whenever I is a finite non-empty interval with endpoints $a < b$, then

$$\mathbb{P}^x(X_{\tau_I} = a) = \frac{b-x}{b-a}, \qquad \mathbb{P}^x(X_{\tau_I} = b) = \frac{x-a}{b-a}$$

provided $a < x < b$. A continuous regular strong Markov process on the line on natural scale has speed measure m if for each finite non-empty interval I we have

$$\mathbb{E}^x \tau_I = G_I(x)$$

whenever x is in the interior of I.

It is well known that if $(X, \mathbb{P}^x)$ and $(Y, \mathbb{Q}^x)$ are continuous regular strong Markov processes on the line on natural scale with the same speed measure m, then the law of X under $\mathbb{P}^x$ is equal to the law of Y under $\mathbb{Q}^x$ for each x. In addition, X will be a local martingale under $\mathbb{P}^x$ for each x.

Let W_t be a one-dimensional Brownian motion and let $\{L_t^x\}$ be the jointly continuous local times. If we define

$$\alpha_t = \int L_t^y\, m(dy), \tag{2.3}$$

then α_t will be continuous and strictly increasing. If we let β_t be the inverse of α_t and set

$$X_t^M = x_0 + W_{\beta_t}, \tag{2.4}$$

then X^M will be a continuous regular strong Markov process on natural scale with speed measure m starting at x_0. See the references listed above for a proof, e.g., [6, Theorem 41.9]. We denote the law of X^M started at x_0 by $\mathbb{P}_M^{x_0}$.

If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\mathcal{G}$ a σ-field contained in $\mathcal{F}$, a regular conditional probability $\mathbb{Q}$ for $\mathbb{P}(\cdot \mid \mathcal{G})$ is a map from $\Omega \times \mathcal{F}$ to $[0,1]$ such that
(1) for each $A \in \mathcal{F}$, $\mathbb{Q}(\cdot, A)$ is measurable with respect to $\mathcal{F}$;
(2) for each $\omega \in \Omega$, $\mathbb{Q}(\omega, \cdot)$ is a probability measure on $\mathcal{F}$;
(3) for each $A \in \mathcal{F}$, $\mathbb{P}(A \mid \mathcal{G})(\omega) = \mathbb{Q}(\omega, A)$ for almost every ω.

Regular conditional probabilities do not always exist, but will if Ω has sufficient structure; see [6, Appendix C].

The filtration $\{\mathcal{F}_t\}$ generated by a process Z is the smallest filtration to which Z is adapted and which satisfies the usual conditions.

We use the letter c with or without subscripts to denote finite positive constants whose value may change from place to place.

3 Speed measures for local martingales

Let $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ be Borel measurable functions with $a(x) \le b(x)$ for all x. If S is a finite stopping time, let

$$\tau_{[a,b]}^S = \inf\{t > S : X_t \notin [a(X_S), b(X_S)]\}.$$

We say a continuous local martingale X started at x_0 has speed measure m if $X_0 = x_0$ and

$$\mathbb{E}\,[\tau_{[a,b]}^S - S \mid \mathcal{F}_S] = G_{[a(X_S),b(X_S)]}(X_S), \qquad \text{a.s.} \tag{3.1}$$

whenever S is a finite stopping time and a and b are as above.

Remark 3.1. We remark that if X were a strong Markov process, then the left hand side of (3.1) would be equal to $\mathbb{E}^{X_S} \tau_{[a,b]}^0$, where $\tau_{[a,b]}^0 = \inf\{t \ge 0 : X_t \notin [a,b]\}$. Thus the above definition of speed measure for a martingale is a generalization of the one for one-dimensional diffusions on natural scale.

Theorem 3.2. *Let m be a measure that is finite and positive on every finite open interval. There exists a continuous local martingale X with m as its speed measure.*

Proof. Set X equal to X^M as defined in (2.4). We only need show that (3.1) holds. Since X is a Markov process and has associated with it probabilities $\mathbb{P}^x$ and shift operators θ_t, then

$$\tau^S_{[a,b]} - S = \sigma_{[a(X_0),b(X_0)]} \circ \theta_S,$$

where $\sigma_{[a(X_0),b(X_0)]} = \inf\{t > 0 : X_t \notin [a(X_0), b(X_0)]\}$. By the strong Markov property,

$$\mathbb{E}\,[\tau^S_{[a,b]} - S \mid \mathcal{F}_S] = \mathbb{E}^{\,X_S}\sigma_{[a(X_0),b(X_0)]} \qquad \text{a.s.} \tag{3.2}$$

For each y, $\sigma_{[a(X_0),b(X_0)]} = \tau_{[a(y),b(y)]}$ under $\mathbb{P}^y$, and therefore

$$\mathbb{E}^{\,y}\sigma_{[a(X_0),b(X_0)]} = G_{[a(y),b(y)]}(y).$$

Replacing y by $X_S(\omega)$ and substituting in (3.2) yields (3.1). □

Theorem 3.3. *Let X be any continuous local martingale that has speed measure m and let f be a non-negative Borel measurable function. Suppose $X_0 = x_0$, a.s. Let $I = [a, b]$ be a finite interval with $a < b$ such that m does not give positive mass to either end point. Then*

$$\mathbb{E}\int_0^{\tau_I} f(X_s)\,ds = \int_I g_I(x,y)f(y)\,m(dy). \tag{3.3}$$

Proof. It suffices to suppose that f is continuous and equal to 0 at the boundaries of I and then to approximate an arbitrary non-negative Borel measurable function by continuous functions that are 0 on the boundaries of I. The main step is to prove

$$\mathbb{E}\int_0^{\tau_I(X)} f(X_s)\,ds = \mathbb{E}\int_0^{\tau_I(X^M)} f(X^M_s)\,ds. \tag{3.4}$$

Let $\varepsilon > 0$. Choose δ such that $|f(x) - f(y)| < \varepsilon$ if $|x - y| < \delta$ with $x, y \in I$.

Set $S_0 = 0$ and

$$S_{i+1} = \inf\{t > S_i : |X_t - X_{S_i}| \geq \delta\}.$$

Then

$$\mathbb{E}\int_0^{\tau_I} f(X_s)\,ds = \mathbb{E}\sum_{i=0}^{\infty}\int_{S_i\wedge\tau_I}^{S_{i+1}\wedge\tau_I} f(X_s)\,ds$$

differs by at most $\varepsilon\mathbb{E}\,\tau_I$ from

$$\begin{aligned}\mathbb{E}\sum_{i=0}^{\infty} f(X_{S_i\wedge\tau_I})(S_{i+1}\wedge\tau_I - S_i\wedge\tau_I) \qquad\qquad (3.5)\\ = \mathbb{E}\Big[\sum_{i=0}^{\infty} f(X_{S_i\wedge\tau_I})\mathbb{E}\left[S_{i+1}\wedge\tau_I - S_i\wedge\tau_I \mid \mathcal{F}_{S_i\wedge\tau_I}\right]\Big].\end{aligned}$$

Let $a(x) = a\vee(x-\delta)$ and $b(x) = b\wedge(x+\delta)$. Since X is a continuous local martingale with speed measure m, the last line in (3.5) is equal to

$$\mathbb{E}\sum_{i=0}^{\infty} f(X_{S_i\wedge\tau_I})G_{[a(X_{S_i\wedge\tau_I}),b(X_{S_i\wedge\tau_I})]}(X_{S_i\wedge\tau_I}). \qquad (3.6)$$

Because $\mathbb{E}\,\tau_{[-N,N]}<\infty$ for all N, then X is a time change of a Brownian motion. It follows that the distribution of $\{X_{S_i\wedge\tau_I(X)}, i\geq 0\}$ is that of a simple random walk on the lattice $\{x+k\delta\}$ stopped the first time it exits I, and thus is the same as the distribution of $\{X^M_{S_i\wedge\tau_I(X^M)}, i\geq 0\}$. Therefore the expression is (3.6) is equal to the corresponding expression with X replaced by X^M. This in turns differs by at most $\mathbb{E}\,\varepsilon\tau_I(X^M)$ from

$$\mathbb{E}\int_0^{\tau_I(X^M)} f(X^M_s)\,ds.$$

Since ε is arbitrary, we have (3.4). Finally, the right hand side of (3.4) is equal to the right hand side of (3.3) by [5, Corollary IV.2.4]. □

4 Uniqueness in law

In this section we show that if X is a continuous local martingale under $\mathbb{P}$ with speed measure m, then X has the same law as X^M. Note that we do not suppose *a priori* that X is a strong Markov process. We remark that the results of [12] do not apply, since in that paper a generalization of the system (1.5)–(1.6) is studied rather than the formulation given by (1.5) together with (1.7).

Theorem 4.1. *Suppose $\mathbb{P}$ is a probability measure and X is a continuous local martingale with respect to $\mathbb{P}$. Suppose that X has speed measure m and $X_0 = x_0$ a.s. Then the law of X under $\mathbb{P}$ is equal to the law of X^M under $\mathbb{P}_M^{x_0}$.*

Proof. Let $R > 0$ be such that $m(\{-R\}) = m(\{R\}) = 0$ and set $I = [-R, R]$. Let $\overline{X}_t = X_{t\wedge\tau_I(X)}$ and $\overline{X}_t^M = X^M_{t\wedge\tau_I(X^M)}$, the processes X and X^M stopped on exiting I. For f bounded and measurable let

$$H_\lambda f = \mathbb{E} \int_0^{\tau_I(\overline{X})} e^{-\lambda t} f(\overline{X}_t)\, dt$$

and

$$H_\lambda^M f(x) = \mathbb{E}^{\,x} \int_0^{\tau_I(\overline{X}^M)} e^{-\lambda t} f(\overline{X}_t^M)\, dt$$

for $\lambda \geq 0$. Since $\overline{X}$ and $\overline{X}^M$ are stopped at times $\tau_I(\overline{X})$ and $\tau_I(\overline{X}^M)$, resp., we can replace $\tau_I(\overline{X})$ and $\tau_I(\overline{X}^M)$ by ∞ in both of the above integrals without affecting H_λ or H_λ^M as long as f is 0 on the boundary of I.

Suppose $f(-R) = f(R) = 0$. Then $H_\lambda^M f(-R)$ and $H_\lambda^M f(R)$ are also 0, since we are working with the stopped process.

We want to show

$$H_\lambda f = H_\lambda^M f(x_0), \qquad \lambda \geq 0. \tag{4.1}$$

By Theorem 3.3 we know (4.1) holds for $\lambda = 0$. Let $K = \mathbb{E}\,\tau_I(X)$. We have $\mathbb{E}^{\,x_0}\tau_I(X^M) = K$ as well since both X and X^M have speed measure m.

Let $\lambda = 0$ and $\mu \leq 1/2K$. Let $t > 0$ and let $Y_s = \overline{X}_{s+t}$. Let $\mathbb{Q}_t$ be a regular conditional probability for $\mathbb{P}(Y \in \cdot \mid \mathcal{F}_t)$. It is easy to see that for almost every ω, Y is a continuous local martingale under $\mathbb{Q}_t(\omega, \cdot)$ started at $\overline{X}_t$ and Y has speed measure m. Cf. [5, Section I.5] or [7]. Therefore by Theorem 3.3

$$\mathbb{E}_{\mathbb{Q}_t} \int_0^\infty f(Y_s)\, ds = H_0^M f(\overline{X}_t).$$

This can be rewritten as

$$\mathbb{E}\Big[\int_0^\infty f(\overline{X}_{s+t})\, ds \mid \mathcal{F}_t\Big] = H_0^M f(\overline{X}_t), \qquad \text{a.s.} \tag{4.2}$$

as long as f is 0 on the endpoints of I.

Therefore, recalling that $\lambda = 0$,

$$\begin{aligned} H_\mu H^M_\lambda f &= \mathbb{E} \int_0^\infty e^{-\mu t} H^M_\lambda f(\overline{X}_t)\, dt \qquad (4.3)\\ &= \mathbb{E} \int_0^\infty e^{-\mu t} \mathbb{E}\Big[\int_0^\infty e^{-\lambda s} f(\overline{X}_{s+t})\, ds \mid \mathcal{F}_t \Big]\, dt \\ &= \mathbb{E} \int_0^\infty e^{-\mu t} e^{\lambda t} \int_t^\infty e^{-\lambda s} f(\overline{X}_s)\, ds\, dt \\ &= \mathbb{E} \int_0^\infty \int_0^s e^{-(\mu-\lambda)t}\, dt\, e^{-\lambda s} f(\overline{X}_s)\, ds \\ &= \mathbb{E} \int_0^\infty \frac{1 - e^{-(\mu-\lambda)s}}{\mu - \lambda} e^{-\lambda s} f(\overline{X}_s)\, ds \\ &= \frac{1}{\mu-\lambda} \mathbb{E} \int_0^\infty e^{-\lambda s} f(\overline{X}_s)\, ds - \frac{1}{\mu-\lambda} \mathbb{E} \int_0^\infty e^{-\mu s} f(\overline{X}_s)\, ds. \\ &= \frac{1}{\mu-\lambda} H^M_\lambda f(x_0) - \frac{1}{\mu-\lambda} \mathbb{E} \int_0^\infty e^{-\mu s} f(\overline{X}_s)\, ds. \end{aligned}$$

We used (4.2) in the second equality. Rearranging,

$$H_\mu f = H^M_\lambda f(x_0) + (\lambda - \mu) H_\mu (H^M_\lambda f). \qquad (4.4)$$

Since $\overline{X}$ and $\overline{X}^M$ are stopped upon exiting I, then $H^M_\lambda f = 0$ at the endpoints of I. We now take (4.4) with f replaced by $H^M_\lambda f$, use this to evaluate the last term in (4.4), and obtain

$$H_\mu f = H^M_\lambda f(x_0) + (\lambda - \mu) H^M_\lambda (H^M_\lambda f)(x_0) + (\lambda - \mu)^2 H_\mu (H^M_\lambda (H^M_\lambda f)).$$

We continue. Since

$$|H_\mu g| \le \|g\| \mathbb{E}\, \tau_I(X) = \|g\| K$$

and

$$\|H^M_\lambda g\| \le \|g\| \mathbb{E}\, \tau_I(X^M) = \|g\| K$$

for each bounded g, where $\|g\|$ is the supremum norm of g, we can iterate and get convergence as long as $\mu \le 1/2K$ and obtain

$$H_\mu f = H^M_\lambda f(x_0) + \sum_{i=1}^\infty ((\lambda - \mu) H^M_\lambda)^i H^M_\lambda f(x_0).$$

The above also holds when $\overline{X}$ is replaced by $\overline{X}^M$, so that

$$H_\mu^M f(x_0) = H_\lambda^M f(x_0) + \sum_{i=1}^\infty ((\lambda-\mu)H_\lambda^M)^i H_\lambda^M f(x_0).$$

We conclude $H_\mu f = H_\mu^M f(x_0)$ as long as $\mu \le 1/2K$ and f is 0 on the endpoints of I.

This holds for every starting point. If $Y_s = \overline{X}_{s+t}$ and $\mathbb{Q}_t$ is a regular conditional probability for the law of Y_s under $\mathbb{P}^x$ given $\mathcal{F}_t$, then we asserted above that Y is a continuous local martingale started at $\overline{X}_t$ with speed measure m under $\mathbb{Q}_t(\omega,\cdot)$ for almost every ω. We replace x_0 by $\overline{X}_t(\omega)$ in the preceding paragraph and derive

$$\mathbb{E}\Big[\int_0^\infty e^{-\mu s} f(\overline{X}_{s+t})\,ds \mid \mathcal{F}_t\Big] = H_\mu^M f(\overline{X}_t), \qquad \text{a.s.}$$

if $\mu \le 1/2K$ and f is 0 on the endpoints of I.

We now take $\lambda = 1/2K$ and $\mu \in (1/2K, 2/2K]$. The same argument as above shows that $H_\mu f = H_\mu^M f(x_0)$ as long as f is 0 on the endpoints of I. This is true for every starting point. We continue, letting $\lambda = n/2K$ and using induction, and obtain

$$H_\mu f = H_\mu^M f(x_0)$$

for every $\mu \ge 0$.

Now suppose f is continuous with compact support and R is large enough so that $(-R,R)$ contains the support of f. We have that

$$\mathbb{E}\int_0^{\tau_{[-R,R]}(\overline{X})} e^{-\mu t} f(\overline{X}_t)\,dt = \mathbb{E}^{x_0}\int_0^{\tau_{[-R,R]}(\overline{X}^M)} e^{-\mu t} f(\overline{X}_t^M)\,dt$$

for all $\mu > 0$. This can be rewritten as

$$\mathbb{E}\int_0^\infty e^{-\mu t} f(X_{t\wedge \tau_{[-R,R]}(X)})\,dt = \mathbb{E}^{x_0}\int_0^\infty e^{-\mu t} f(X^M_{t\wedge\tau_{[R,R]}(X^M)})\,dt. \qquad (4.5)$$

If we hold μ fixed and let $R\to\infty$ in (4.5), we obtain

$$\mathbb{E}\int_0^\infty e^{-\mu t} f(X_t)\,dt = \mathbb{E}^{x_0}\int_0^\infty e^{-\mu t} f(X_t^M)\,dt$$

for all $\mu > 0$. By the uniqueness of the Laplace transform and the continuity of f, X, and X^M,

$$\mathbb{E}\, f(X_t) = \mathbb{E}^{\,x_0} f(X_t^M)$$

for all t. By a limit argument, this holds whenever f is a bounded Borel measurable function.

The starting point x_0 was arbitrary. Using regular conditional probabilities as above,

$$\mathbb{E}\,[f(X_{t+s}) \mid \mathcal{F}_t] = \mathbb{E}^{\,x_0}[f(X_{t+s}^M) \mid \mathcal{F}_t].$$

By the Markov property, the right hand side is equal to

$$\mathbb{E}^{\,X_t^M} f(X_s) = P_s f(X_t^M),$$

where P_s is the transition probability kernel for X^M.

To prove that the finite dimensional distributions of X and X^M agree, we use induction. We have

$$\begin{aligned}\mathbb{E} \prod_{j=1}^{n+1} f_j(X_{t_j}) &= \mathbb{E}_{\,i} \prod_{j=1}^{n} f_j(X_{t_j}) \mathbb{E}_{\,i}[f_{n+1}(X_{t_{n+1}}) \mid \mathcal{F}_{t_n}] \\ &= \mathbb{E}_{\,i} \prod_{j=1}^{n} f_j(X_{t_j}) P_{t_{n+1}-t_n} f_{n+1}(X_{t_n}).\end{aligned}$$

We use the induction hypothesis to see that this is equal to

$$\mathbb{E}^{\,x_0} \prod_{j=1}^{n} f_j(X_{t_j}^M) P_{t_{n+1}-t_n} f_{n+1}(X_{t_n}^M).$$

We then use the Markov property to see that this in turn is equal to

$$\mathbb{E}^{\,x_0} \prod_{j=1}^{n+1} f_j(X_{t_j}^M).$$

Since X and X^M have continuous paths and the same finite dimensional distributions, they have the same law. □

5 The stochastic differential equation

We now discuss the particular stochastic differential equation we want our martingales to solve. We specialize to the following speed measure. Let $\gamma \in (0, \infty)$ and let

$$m(dx) = dx + \gamma\delta_0(dx), \tag{5.1}$$

where δ_0 is point mass at 0.

We consider the stochastic differential equation

$$X_t = x_0 + \int_0^t 1_{(X_s \neq 0)}\, dW_s. \tag{5.2}$$

A triple $(X, W, \mathbb{P})$ is a weak solution to (5.2) with X starting at x_0 if $\mathbb{P}$ is a probability measure, there exists a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions, W is a Brownian motion under $\mathbb{P}$ with respect to $\{\mathcal{F}_t\}$, and X is a continuous martingale adapted to $\{\mathcal{F}_t\}$ with $X_0 = x_0$ and satisfying (5.2).

We now show that any martingale with $X_0 = x_0$ a.s. that has speed measure m is the first element of a triple that is a weak solution to (5.2). Although X has the same law as X^M started at x_0, here we only have one probability measure and we cannot assert that X is a strong Markov process. We point out that [12, Theorem 5.18] does not apply here, since they study a generalization of the system (1.5)–(1.6), and we do not know at this stage that this formulation is equivalent to the one used here.

Theorem 5.1. *Let $\mathbb{P}$ be a probability measure on a space that supports a Brownian motion and let X be a continuous martingale which has speed measure m with $X_0 = x_0$ a.s. Then there exists a Brownian motion W such that $(X, W, \mathbb{P})$ is a weak solution to (5.2) with X starting at x_0. Moreover*

$$X_t = x_0 + \int_0^t 1_{(X_s \neq 0)}\, dX_s. \tag{5.3}$$

Proof. Let

$$W'_t = \int_0^t 1_{(X_s \neq 0)}\, dX_s.$$

Hence

$$d\langle W' \rangle_t = 1_{(X_t \neq 0)}\, d\langle X \rangle_t.$$

Let $0 < \eta < \delta$. Let $S_0 = \inf\{t : |X_t| \geq \delta\}$, $T_i = \inf\{t > S_i : |X_t| \leq \eta\}$, and $S_{i+1} = \inf\{t > T_i : |X_t| \geq \delta\}$ for $i = 0, 1, \ldots$.

The speed measure of X is equal to m, which in turn is equal to Lebesgue measure on $\mathbb{R} \setminus \{0\}$, hence X has the same law as X^M by Theorem 4.1. Since X^M behaves like a Brownian motion when it is away from zero, we conclude $1_{[S_i,T_i]}\, d\langle X\rangle_t = 1_{[S_i,T_i]}\, dt$.

Thus for each N,

$$\int_0^t 1_{\cup_{i=0}^N [S_i,T_i]}(s)\, d\langle X\rangle_s = \int_0^t 1_{\cup_{i=0}^N [S_i,T_i]}(s)\, ds.$$

Letting $N \to \infty$, then $\eta \to 0$, and finally $\delta \to \infty$, we obtain

$$\int_0^t 1_{(X_s \neq 0)}\, d\langle X\rangle_s = \int_0^t 1_{(X_s \neq 0)}\, ds.$$

Let V_t be an independent Brownian motion and let

$$W''_t = \int_0^t 1_{(X_s = 0)}\, dV_s.$$

Let $W_t = W'_t + W''_t$. Clearly W' and W'' are orthogonal martingales, so

$$d\langle W\rangle_t = d\langle W'\rangle_t + d\langle W''\rangle_t = 1_{(X_t \neq 0)}\, dt + 1_{(X_t = 0)}\, dt = dt.$$

By Lévy's theorem (see [6, Theorem 12.1]), W is a Brownian motion.

If

$$M_t = \int_0^t 1_{(X_s = 0)}\, dX_s,$$

by the occupation times formula ([22, Corollary VI.1.6]),

$$\langle M\rangle_t = \int_0^t 1_{(X_s = 0)}\, d\langle X\rangle_s = \int 1_{\{0\}}(x) \ell_t^x(X)\, dx = 0$$

for all t, where $\{\ell_t^x(X)\}$ are the local times of X in the semimartingale sense. This implies that M_t is identically zero, and hence $X_t = W'_t$.

Using the definition of W, we deduce

$$1_{(X_t \neq 0)}\, dW_t = 1_{(X_t \neq 0)}\, dX_t = dW'_t = dX_t, \tag{5.4}$$

as required. □

We now show weak uniqueness, that is, if $(X, W, \mathbb{P})$ and $(\widetilde{X}, \widetilde{W}, \widetilde{\mathbb{P}})$ are two weak solutions to (5.2) with X and $\widetilde{X}$ starting at x_0 and in addition X and $\widetilde{X}$ have speed measure m, then the joint law of (X, W) under $\mathbb{P}$ equals the joint law of $(\widetilde{X}, \widetilde{W})$ under $\widetilde{\mathbb{P}}$. This holds even though W will not in general be adapted to the filtration of X. We know that the law of X under $\mathbb{P}$ equals the law of $\widetilde{X}$ under $\widetilde{\mathbb{P}}$ and also that the law of W under $\mathbb{P}$ equals the law of $\widetilde{W}$ under $\widetilde{\mathbb{P}}$, but the issue here is the joint law. Cf. [8]. See also [11].

Theorem 5.2. *Suppose $(X, W, \mathbb{P})$ and $(\widetilde{X}, \widetilde{W}, \widetilde{\mathbb{P}})$ are two weak solutions to (5.2) with $X_0 = \widetilde{X}_0 = x_0$ and that X and $\widetilde{X}$ are both continuous martingales with speed measure m. Then the joint law of (X, W) under $\mathbb{P}$ equals the joint law of $(\widetilde{X}, \widetilde{W})$ under $\widetilde{\mathbb{P}}$.*

Proof. Recall the construction of X^M from Section 2. With U_t a Brownian motion with jointly continuous local times $\{L_t^x\}$ and m given by (5.1), we define α_t by (2.3), let β_t be the right continuous inverse of α_t, and let $X_t^M = x_0 + U_{\beta_t}$. Since m is greater than or equal to Lebesgue measure but is finite on every finite interval, we see that α_t is strictly increasing, continuous, and $\lim_{t\to\infty} \alpha_t = \infty$. It follows that β_t is continuous and tends to infinity almost surely as $t \to \infty$.

Given any stochastic process $\{N_t, t \geq 0\}$, let $\mathcal{F}_\infty^N$ be the σ-field generated by the collection of random variables $\{N_t, t \geq 0\}$ together with the null sets.

We have $\beta_t = \langle X^M \rangle_t$ and $U_t = X_{\alpha_t}^M - x_0$. Since β_t is measurable with respect to $\mathcal{F}_\infty^{X^M}$ for each t, then α_t is also, and hence so is U_t. In fact, we can give a recipe to construct a Borel measurable map $F : C[0, \infty) \to C[0, \infty)$ such that $U = F(X^M)$. Note also that X_t^M is measurable with respect to $\mathcal{F}_\infty^U$ for each t and there exists a Borel measurable map $G : C[0, \infty) \to C[0, \infty)$ such that $X^M = G(U)$. In addition observe that $\langle X^M \rangle_\infty = \infty$ a.s.

Since X and X^M have the same law, then $\langle X \rangle_\infty = \infty$ a.s. If Z_t is a Brownian motion with $X_t = x_0 + Z(\zeta_t)$ for a continuous increasing process ζ, then $\zeta_t = \langle X \rangle_t$ is measurable with respect to $\mathcal{F}_\infty^X$, its inverse ρ_t is also, and therefore $Z_t = X_{\rho_t} - x_0$ is as well. Moreover the recipe for constructing Z from X is exactly the same as the one for constructing U from X^M, that is, $Z = F(X)$. Since X and X^M have the same law, then the joint law of (X, Z) is equal to the joint law of (X^M, U). We can therefore conclude that X is measurable with respect to $\mathcal{F}_\infty^Z$ and $X = G(Z)$.

Let

$$Y_t = \int_0^t 1_{(X_s=0)}\, dW_s.$$

Then Y is a martingale with

$$\langle Y \rangle_t = \int_0^t 1_{(X_s=0)}\, ds = t - \langle X \rangle_t.$$

Observe that $\langle X, Y \rangle_t = \int_0^t 1_{(X_s \neq 0)} 1_{(X_s=0)}\, ds = 0$. By a theorem of Knight (see [17] or [22]), there exists a two-dimensional process $V = (V_1, V_2)$ such that V is a two-dimensional Brownian motion under $\mathbb{P}$ and

$$(X_t, Y_t) = (x_0 + V_1(\langle X \rangle_t), V_2(\langle Y \rangle_t)), \qquad \text{a.s.}$$

(It turns out that $\langle Y \rangle_\infty = \infty$, but that is not needed in Knight's theorem.)

By the third paragraph of this proof, $X_t = x_0 + V_1(\langle X \rangle_t)$ implies that X_t is measurable with respect to $\mathcal{F}_\infty^{V_1}$, and in fact $X = G(V_1)$. Since $\langle Y \rangle_t = t - \langle X \rangle_t$, then (X_t, Y_t) is measurable with respect to $\mathcal{F}_\infty^V$ for each t and there exists a Borel measurable map $H : C([0,\infty), \mathbb{R}^2) \to C([0,\infty), \mathbb{R}^2)$, where $C([0,\infty), \mathbb{R}^2)$ is the space of continuous functions from $[0,\infty)$ to $\mathbb{R}^2$, and $(X, Y) = H(V)$. Thus (X, Y) is the image under H of a two-dimensional Brownian motion. If $(\widetilde{X}, \widetilde{W}, \widetilde{\mathbb{P}})$ is another weak solution, then we can define $\widetilde{Y}$ analogously and find a two-dimensional Brownian motion $\widetilde{V}$ such that $(\widetilde{X}, \widetilde{Y}) = H(\widetilde{V})$. The key point is that the same H can be used. We conclude that the law of (X, Y) is uniquely determined. Since

$$(X, W) = (X, X + Y - x_0),$$

this proves that the joint law of (X, W) is uniquely determined. □

Remark 5.3. In Section 2 we constructed the continuous strong Markov process $(X^M, \mathbb{P}_M^x)$ and we now know that X started at x_0 is equal in law to X^M under $\mathbb{P}_M^{x_0}$. We pointed out in Remark 3.1 that in the strong Markov case the notion of speed measure for a martingale reduces to that of speed measure for a one dimensional diffusion. In [11] it is shown that the solution to the system (1.5)–(1.6) is unique in law and thus the solution started at x_0 is equal in law to that of a diffusion on $\mathbb{R}$ started at x_0; let $\widetilde{m}$ be the speed measure for this strong Markov process. Thus to show the equivalence of the system (1.5)–(1.6) to the one given by (1.5) and (1.7), it suffices to show that $\widetilde{m} = m$ if and only if (1.6) holds, where m is given by (5.1) and $\gamma = 1/\mu$. Clearly both $\widetilde{m}$ and m are equal to Lebesgue measure on $\mathbb{R} \setminus \{0\}$, so it suffices to compare the atoms of $\widetilde{m}$ and m at 0.

Suppose (1.6) holds and $\gamma = 1/\mu$. Let $A_t = \int_0^t 1_{\{0\}}(X_s)\, ds$. Thus (1.6) asserts that $A_t = \frac{1}{\mu}\ell_t^0$. Let $I = [a, b] = [-1, 1]$, $x_0 = 0$, and τ_I the first time that X leaves the interval I. Setting $t = \tau_I$ and taking expectations starting from 0, we have

$$\mathbb{E}^{\,0} A_{\tau_I} = \frac{1}{\mu}\mathbb{E}^{\,0}\ell^0_{\tau_I}.$$

Since ℓ_t^0 is the increasing part of the submartingale $|X_t - x_0| - |x_0|$ and X_{τ_I} is equal to either 1 or -1, the right hand side is equal to

$$\frac{1}{\mu}\mathbb{E}^{\,0}|X_{\tau_I}| = \frac{1}{\mu}.$$

On the other hand, by [5, (IV.2.11)],

$$\mathbb{E}^{\,0} A_{\tau_I} = \int_{-1}^{1} g_I(0, y) 1_{\{0\}}(y)\, \widetilde{m}(dy) = \widetilde{m}(\{0\}).$$

Thus $\widetilde{m} = m$ if $\gamma = 1/\mu$.

Now suppose we have a solution to the pair (1.5) and (1.7) and $\gamma = 1/\mu$; we will show (1.6) holds. Let $R > 0$, $I = [-R, R]$, and τ_I the first exit time from I. Set $B_t = \frac{1}{\mu}\ell_t^0$. For any $x \in I$, we have by [5, (IV.2.11)] that

$$\mathbb{E}^{\,x} A_{\tau_I} = \int_{-1}^{1} g_I(x, y) 1_{\{0\}}(y)\, m(dy) = \gamma g_I(x, 0). \tag{5.5}$$

Taking expectations,

$$\mathbb{E}^{\,x} B_{\tau_I} = \frac{1}{\mu}\mathbb{E}^{\,x}[\, |X_{\tau_I} - x| - |x|\,]. \tag{5.6}$$

Since X is a time change of a Brownian motion that exits I a.s., the distribution of X_{τ_I} started at x is the same as that of a Brownian motion started at x upon exiting I. A simple computation shows that the right hand side of (5.6) agrees with the right hand side of (5.5). By the strong Markov property,

$$\mathbb{E}^{\,0}[A_{\tau_I} - A_{\tau_I \wedge t} \mid \mathcal{F}_t] = \mathbb{E}^{\,X_t} A_{\tau_I} = \mathbb{E}^{\,X_t} B_{\tau_I} = \mathbb{E}^{\,0}[B_{\tau_I} - B_{\tau_I \wedge t} \mid \mathcal{F}_t]$$

almost surely on the set $(t \le \tau_I)$. Observe that if $U_t = \mathbb{E}^0[A_{\tau_I} - A_{\tau_I\wedge t} \mid \mathcal{F}_t]$, then we can write

$$U_t = \mathbb{E}^0[A_{\tau_I} - A_{\tau_I\wedge t} \mid \mathcal{F}_t] = \mathbb{E}^0[A_{\tau_I} \mid \mathcal{F}_t] - A_{\tau_I\wedge t}$$

and

$$U_t = \mathbb{E}^0[B_{\tau_I} - B_{\tau_I\wedge t} \mid \mathcal{F}_t] = \mathbb{E}^0[B_{\tau_I} \mid \mathcal{F}_t] - B_{\tau_I\wedge t}$$

for $t \le \tau_I$. This expresses the supermartingale U as a martingale minus an increasing process in two different ways. By the uniqueness of the Doob decomposition for supermartingales, we conclude $A_{\tau_I\wedge t} = B_{\tau_I\wedge t}$ for $t \le \tau_I$. Since R is arbitrary, this establishes (1.6). (The argument that the potential of an increasing process determines the process is well known.)

Remark 5.4. In the remainder of the paper we prove that there does not exist a strong solution to the pair (1.5) and (1.7) nor does pathwise uniqueness hold. In [11], the authors prove that there is no strong solution to the pair (1.5) and (1.6) and that pathwise uniqueness does not hold. Since we now know there is an equivalence between the pair (1.5) and (1.7) and the pair (1.5) and (1.6), one could at this point use the argument of [11] in place of the argument of this paper. Alternatively, in the paper of [11] one could use our argument in place of theirs to establish the non-existence of a strong solution and that pathwise uniqueness does not hold.

6 Approximating processes

Let $\widetilde{W}$ be a Brownian motion adapted to a filtration $\{\mathcal{F}_t, t \ge 0\}$, let $\varepsilon \le \gamma$, and let X_t^ε be the solution to

$$dX_t^\varepsilon = \sigma_\varepsilon(X_t^\varepsilon)\, d\widetilde{W}_t, \qquad X_0^\varepsilon = x_0, \tag{6.1}$$

where

$$\sigma_\varepsilon(x) = \begin{cases} 1, & |x| > \varepsilon; \\ \sqrt{\varepsilon/\gamma}, & |x| \le \varepsilon. \end{cases}$$

For each x_0 the solution to the stochastic differential equation is pathwise unique by [20] or [21]. We also know that if $\mathbb{P}_\varepsilon^x$ is the law of X^ε starting from

x, then $(X^\varepsilon, \mathbb{P}^x_\varepsilon)$ is a continuous regular strong Markov process on natural scale. The speed measure of X^ε will be

$$m_\varepsilon(dy) = dy + \frac{\gamma}{\varepsilon} 1_{[-\varepsilon,\varepsilon]}(y)\, dy.$$

Let Y^ε be the solution to

$$dY^\varepsilon_t = \sigma_{2\varepsilon}(Y^\varepsilon_t)\, d\widetilde{W}_t, \qquad Y^\varepsilon_0 = x_0. \tag{6.2}$$

Since $\sigma_\varepsilon \le 1$, then $d\langle X^\varepsilon\rangle_t \le dt$. By the Burkholder-Davis-Gundy inequalities (see, e.g., [6, Section 12.5]),

$$\mathbb{E}\, |X^\varepsilon_t - X^\varepsilon_s|^{2p} \le c|t-s|^p \tag{6.3}$$

for each $p \ge 1$, where the constant c depends on p. It follows (for example, by Theorems 8.1 and 32.1 of [6]) that the law of X^ε is tight in $C[0, t_0]$ for each t_0. The same is of course true for Y^ε and $\widetilde{W}$, and so the triple $(X^\varepsilon, Y^\varepsilon, \widetilde{W})$ is tight in $(C[0, t_0])^3$ for each $t_0 > 0$.

Let P^ε_t be the transition probabilities for the Markov process X^ε. Let C_0 be the set of continuous functions on $\mathbb{R}$ that vanish at infinity and let

$$L = \{f \in C_0 : |f(x) - f(y)| \le |x-y|, x, y \in \mathbb{R}\},$$

the set of Lipschitz functions with Lipschitz constant 1 that vanish at infinity.

One of the main results of [3] (see Theorem 4.2) is that P^ε_t maps L into L for each t and each $\varepsilon < 1$.

Theorem 6.1. *If $f \in C_0$, then $P^\varepsilon_t f$ converges uniformly for each $t \ge 0$. If we denote the limit by $P_t f$, then $\{P_t\}$ is a family of transition probabilities for a continuous regular strong Markov process $(X, \mathbb{P}^x)$ on natural scale with speed measure given by* (5.1)*. For each x, $\mathbb{P}^x_\varepsilon$ converges weakly to $\mathbb{P}^x$ with respect to $C[0, N]$ for each N.*

Proof. Step 1. Let $\{g_j\}$ be a countable collection of C^2 functions in L with compact support such that the set of finite linear combinations of elements of $\{g_j\}$ is dense in C_0 with respect to the supremum norm.

Let ε_n be a sequence converging to 0. Suppose g_j has support contained in $[-K, K]$ with $K > 1$. Since X^ε_t is a Brownian motion outside $[-1, 1]$, if $|x| > 2K$, then

$$|P^\varepsilon_t g_j(x)| = |\mathbb{E}^{\,x} g_j(X^\varepsilon_t)| \le \|g_j\|\, \mathbb{P}^x(|X^\varepsilon| \text{ hits } |x|/2 \text{ before time } t),$$

which tends to 0 uniformly over $\varepsilon < 1$ as $|x| \to \infty$. Here $\|g_j\|$ is the supremum norm of g_j. By the equicontinuity of the $P_t^\varepsilon g_j$, using the diagonalization method there exists a subsequence, which we continue to denote by ε_n, such that $P_t^{\varepsilon_n} g_j$ converges uniformly on $\mathbb{R}$ for every rational $t \geq 0$ and every j. We denote the limit by $P_t g_j$.

Since $g_j \in C^2$,

$$\begin{aligned} P_t^\varepsilon g_j(x) - P_s^\varepsilon g_j(x) &= \mathbb{E}^{\,x} g_j(X_t^\varepsilon) - \mathbb{E}^{\,x} g_j(X_s^\varepsilon) \\ &= \mathbb{E}^{\,x} \int_s^t \sigma_\varepsilon(X_r^\varepsilon) g_j'(X_r^\varepsilon)\, d\widetilde{W}_r + \tfrac{1}{2}\mathbb{E}^{\,x} \int_s^t \sigma_\varepsilon(X_r^\varepsilon)^2 g_j''(X_r^\varepsilon)\, dr \\ &= \tfrac{1}{2}\mathbb{E}^{\,x} \int_s^t \sigma_\varepsilon(X_r^\varepsilon)^2 g_j''(X_r^\varepsilon)\, dr, \end{aligned}$$

where we used Ito's formula. Since σ_ε is bounded by 1, we obtain

$$|P_t^\varepsilon g_j(x) - P_s^\varepsilon g_j(x)| \leq c_j |t-s|,$$

where the constant c_j depends on g_j. With this fact, we can deduce that $P_t^{\varepsilon_n} g_j$ converges uniformly in C_0 for every $t \geq 0$. We again call the limit $P_t g_j$. Since linear combinations of the g_j's are dense in C_0, we conclude that $P_t^{\varepsilon_n} g$ converges uniformly to a limit, which we call $P_t g$, whenever $g \in C_0$. We note that P_t maps C_0 into C_0.

Step 2. Each X_t^ε is a Markov process, so $P_s^\varepsilon(P_t^\varepsilon g) = P_{s+t}^\varepsilon g$. By the uniform convergence and equicontinuity and the fact that P_s^ε is a contraction, we see that $P_s(P_t g) = P_{s+t} g$ whenever $g \in C_0$.

Let $s_1 < s_2 < \cdots s_j$ and let $f_1, \ldots f_j$ be elements of L. Define inductively $g_j = f_j$, $g_{j-1} = f_{j-1}(P_{s_j - s_{j-1}} g_j)$, $g_{j-2} = f_{j-2}(P_{s_{j-1}-s_{j-2}} g_{j-1})$, and so on. Define g_j^ε analogously where we replace P_t by P_t^ε. By the Markov property applied repeatedly,

$$\mathbb{E}^{\,x}[f_1(X_{s_1}^\varepsilon) \cdots f_j(X_{s_j}^\varepsilon)] = P_{s_1}^\varepsilon g_1^\varepsilon(x).$$

Suppose x is fixed for the moment and let $f_1, \cdots, f_j \in L$. Suppose there is a subsequence $\varepsilon_{n'}$ of ε_n such that $X^{\varepsilon_{n'}}$ converges weakly, say to X, and let $\mathbb{P}'$ be the limit law with corresponding expectation $\mathbb{E}'$. Using the uniform convergence, the equicontinuity, and the fact that P_t^ε maps L into L, we obtain

$$\mathbb{E}'[f_1(X_{s_1}) \cdots f_j(X_{s_j})] = P_{s_1} g_1(x). \tag{6.4}$$

We can conclude several things from this. First, since the limit is the same no matter what subsequence $\{\varepsilon_{n'}\}$ we use, then the full sequence $\mathbb{P}^x_{\varepsilon_n}$ converges weakly. This holds for each starting point x.

Secondly, if we denote the weak limit of the $\mathbb{P}^x_{\varepsilon_n}$ by $\mathbb{P}^x$, then (6.4) holds with $\mathbb{E}'$ replaced by $\mathbb{E}^x$. From this we deduce that $(X, \mathbb{P}^x)$ is a Markov process with transition semigroup given by P_t.

Thirdly, since $\mathbb{P}^x$ is the weak limit of probabilities on $C[0,\infty)$, we conclude that X under $\mathbb{P}^x$ has continuous paths for each x.

Step 3. Since P_t maps C_0 into C_0 and $P_t f(x) = \mathbb{E}^x f(X_t) \to f(x)$ by the continuity of paths if $f \in C_0$, we conclude by [6, Theorem 20.9] that $(X, \mathbb{P}^x)$ is in fact a strong Markov process.

Suppose $f_1, \ldots, f_j$ are in L and $s_1 < s_2 < \cdots < s_j < t < u$. Since X^ε_t is a martingale,

$$\mathbb{E}^x_\varepsilon\Big[X^\varepsilon_u \prod_{i=1}^j f_i(X^\varepsilon_{s_i})\Big] = \mathbb{E}^x\Big[X^\varepsilon_t \prod_{i=1}^j f_i(X^\varepsilon_{s_i})\Big].$$

Moreover, X^ε_t and X^ε_u are uniformly integrable due to (6.3). Passing to the limit along the sequence ε_n, we have the equality with X^ε replaced by X and $\mathbb{E}^x_\varepsilon$ replaced by $\mathbb{E}^x$. Since the collection of random variables of the form $\prod_i f_i(X_{s_i})$ generate $\sigma(X_r; r \le t)$, it follows that X is a martingale under $\mathbb{P}^x$ for each x.

Step 4. Let $\delta, \eta > 0$. Let $I = [q, r]$ and $I^* = [q - \delta, r + \delta]$. In this step we show that

$$\mathbb{E}\,\tau_I(X) = \int_I g_I(0, y)\, m(dy). \tag{6.5}$$

First we obtain a uniform bound on $\tau_{I^*}(X^\varepsilon)$. If $A^\varepsilon_t = t \wedge \tau_{I^*}(X^\varepsilon)$, then

$$\mathbb{E}\,[A^\varepsilon_\infty - A^\varepsilon_t \mid \mathcal{F}_t] = \mathbb{E}^{X^\varepsilon_t} A^\varepsilon_\infty \le \sup_x \mathbb{E}^x \tau_{I^*}(X^\varepsilon).$$

The last term is equal to

$$\sup_x \int_{I^*} g_{I^*}(x, y)\Big(1 + \frac{\gamma}{\varepsilon} 1_{I^*}(y)\Big)\, dy.$$

A simple calculation shows that this is bounded by

$$c(r - q + 2\delta)^2 + c\gamma(r - q + 2\delta),$$

where c does not depend on r, q, δ, or ε. By Theorem I.6.10 of [4], we then deduce that

$$\mathbb{E}\,\tau_{I^*}(X^\varepsilon)^2 = \mathbb{E}\,(A^\varepsilon_\infty)^2 < c < \infty,$$

where c does not depend on ε. By Chebyshev's inequality, for each t,

$$\mathbb{P}(\tau_{I^*}(X^\varepsilon) \geq t) \leq c/t^2.$$

Next we obtain an upper bound on $\mathbb{E}\,\tau_I(X)$ in terms of g_{I^*}. We have

$$\begin{aligned}
\mathbb{P}(\tau_I(X) > t) &= \mathbb{P}(\sup_{s\leq t} |X_s| \leq r, \inf_{s\leq t} |X_s| \geq q)\\
&\leq \limsup_{\varepsilon_n\to 0} \mathbb{P}(\sup_{s\leq t} |X^{\varepsilon_n}_s| \leq r, \inf_{s\leq t} |X_s|^\varepsilon \geq q)\\
&\leq \limsup_{\varepsilon_n\to 0} \mathbb{P}(\tau_{I^*}(X^{\varepsilon_n}) > t) \leq c/t^2.
\end{aligned}$$

Choose u_0 such that

$$\int_{u_0}^\infty \mathbb{P}(\tau_I(X) > t)\,dt < \eta, \qquad \int_{u_0}^\infty \mathbb{P}(\tau_{I^*}(X^{\varepsilon_n}) > t)\,dt < \eta$$

for each ε_n.

Let f and g be continuous functions taking values in $[0, 1]$ such that f is equal to 1 on $(-\infty, r]$ and 0 on $[r + \delta, \infty)$ and g is equal to 1 on $[q, \infty)$ and 0 on $(-\infty, q - \delta]$. We have

$$\begin{aligned}
\mathbb{P}(\sup_{s\leq t} |X_s| \leq r, \inf_{s\leq t} |X_s \geq q) &\leq \mathbb{E}\,[f(\sup_{s\leq t} |X_s|) g(\inf_{s\leq t} |X_s|)]\\
&= \lim_{\varepsilon_n\to 0} \mathbb{E}\,[f(\sup_{s\leq t} |X^{\varepsilon_n}_s|) g(\inf_{s\leq t} |X^{\varepsilon_n}_s|)].
\end{aligned}$$

Then

$$\begin{aligned}
\int_0^{u_0} \mathbb{P}(\tau_I(X) > t)\,dt &= \int_0^{u_0} \mathbb{P}(\sup_{s\leq t} |X_s| \leq r, \inf_{s\leq t} |X_s| \geq q)\,dt\\
&\leq \int_0^{u_0} \mathbb{E}\,[f(\sup_{s\leq t} |X_s|) g(\inf_{s\leq t} |X_s|)]\,dt\\
&= \int_0^{u_0} \lim_{\varepsilon_n\to 0} \mathbb{E}\,[f(\sup_{s\leq t} |X^{\varepsilon_n}_s|) g(\inf_{s\leq t} |X^{\varepsilon_n}_s|)]\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\varepsilon_n \to 0} \int_0^{u_0} \mathbb{E}\,[f(\sup_{s\le t} |X_s^{\varepsilon_n}|) g(\inf_{s\le t} |X_s^{\varepsilon_n}|)]\, dt \\
&\le \limsup_{\varepsilon_n \to 0} \int_0^{u_0} \mathbb{P}(\sup_{s\le t} |X_s^{\varepsilon_n}| \le r+\delta, \inf_{s\le t} |X_s| \ge q-\delta)\, dt \\
&\le \limsup_{\varepsilon_n \to 0} \int_0^{u_0} \mathbb{P}(\tau_{I^*}(X^{\varepsilon_n}) \ge t)\, dt \\
&\le \limsup_{\varepsilon_n \to 0} \mathbb{E}\,\tau_{I^*}(X^{\varepsilon_n}).
\end{aligned}
$$

Hence

$$
\mathbb{E}\,\tau_I(X) \le \int_0^{u_0} \mathbb{P}(\tau_I(X) > t)\, dt + \eta \le \limsup_{\varepsilon_n \to 0} \mathbb{E}\,\tau_{I^*}(X^{\varepsilon_n}) + \eta.
$$

We now use the fact that η is arbitrary and let $\eta \to 0$. Then

$$
\begin{aligned}
\mathbb{E}\,\tau_I(X) &\le \limsup_{\varepsilon_n \to 0} \mathbb{E}\,\tau_{I^*}(X^{\varepsilon_n}) \\
&= \limsup_{\varepsilon_n \to 0} \int_{I^*} g_{I^*}(0,y)\Big(1 + \frac{\gamma}{\varepsilon} 1_{[-\varepsilon,\varepsilon]}(y)\Big)\, dy \\
&= \int_{I^*} g_{I^*}(0,y) m(dy).
\end{aligned}
$$

We next use the joint continuity of $g_{[-a,a]}(x,y)$ in the variables a, x and y. Letting $\delta \to 0$, we obtain

$$
\mathbb{E}\,\tau_I(X) \le \int_I g_I(0,y)\, m(dy).
$$

The lower bound for $\mathbb{E}\,\tau_I(X)$ is done similarly, and we obtain (6.5).

Step 5. Next we show that X is a regular strong Markov process. This means that if $x \ne y$, $\mathbb{P}^x(X_t = y \text{ for some } t) > 0$. To show this, assume without loss of generality that $y < x$. Suppose X starting from x does not hit y with positive probability. Let $z = x + 4|x-y|$. Since $\mathbb{E}^x \tau_{[y,z]} < \infty$, then with probability one X hits z and does so before hitting y. Hence $T_z = \tau_{[y,z]} < \infty$ a.s. Choose t large so that $\mathbb{P}^x(\tau_{[y,z]} > t) < 1/16$. By the optional stopping theorem,

$$
\mathbb{E}^x X_{T_z \wedge t} \ge z\mathbb{P}^x(T_z \le t) + y\mathbb{P}^x(T_z > t) = z - (z-y)\mathbb{P}^x(T_z > t).
$$

By our choice of z, this is greater than x, which contradicts that X is a martingale. Hence X must hit y with positive probability.

Therefore X is a regular continuous strong Markov process on the real line. Since it is a martingale, it is on natural scale. Since its speed measure is the same as that of X^M by (6.5), we conclude from [5, Theorem IV.2.5] that X and X^M have the same law. In particular, X is a martingale with speed measure m.

Step 6. Since we obtain the same limit law no matter what sequence ε_n we started with, the full sequence P_t^ε converges to P_t and $\mathbb{P}_\varepsilon^x$ converges weakly to $\mathbb{P}^x$ for each x.

All of the above applies equally well to Y and its transition probabilities and laws. □

Recall that the sequence $(X^\varepsilon, Y^\varepsilon, \widetilde{W})$ is tight with respect to $(C[0,N])^3$ for each N. Take a subsequence $(X^{\varepsilon_n}, Y^{\varepsilon_n}, \widetilde{W})$ that converges weakly, say to the triple (X, Y, W), with respect to $(C[0,N])^3$ for each N. The last task of this section is to prove that X and Y satisfy (5.2).

Theorem 6.2. *(X, W) and (Y, W) each satisfy* (5.2).

Proof. We prove this for X as the proof for Y is exactly the same. Clearly W is a Brownian motion. Fix N. We will first show

$$\int_0^t 1_{(X_s \neq 0)}\, dX_s = \int_0^t 1_{(X_s \neq 0)}\, dW_s \tag{6.6}$$

if $t \leq N$.

Let $\delta > 0$ and let g be a continuous function taking values in $[0,1]$ such that $g(x) = 0$ if $|x| < \delta$ and $g(x) = 1$ if $|x| \geq 2\delta$. Since g is bounded and continuous and $(X^{\varepsilon_n}, \widetilde{W})$ converges weakly to (X, W), then $(X^{\varepsilon_n}, \widetilde{W}, g(X^{\varepsilon_n}))$ converges weakly to $(X, W, g(X))$. Moreover, since g is 0 on $(-\delta, \delta)$, then

$$\int_0^t g(X_s^{\varepsilon_n})\, d\widetilde{W}_s = \int_0^t g(X_s^{\varepsilon_n})\, dX_s^{\varepsilon_n} \tag{6.7}$$

for ε_n small enough.

By Theorem 2.2 of [19], we have

$$\Big(\int_0^t g(X_s^{\varepsilon_n})\,d\widetilde{W}_s, \int_0^t g(X_s^{\varepsilon_n})\,dX_s^{\varepsilon_n}\Big)$$

converges weakly to

$$\Big(\int_0^t g(X_s)\,dW_s, \int_0^t g(X_s)\,dX_s\Big).$$

Then

$$\begin{aligned}&\mathbb{E}\arctan\Big(\Big|\int_0^t g(X_s)\,dW_s - \int_0^t g(X_s)\,dX_s\Big|\Big)\\ &\quad = \lim_{n\to\infty}\mathbb{E}\arctan\Big(\Big|\int_0^t g(X_s^{\varepsilon_n})\,d\widetilde{W}_s - \int_0^t g(X_s^{\varepsilon_n})\,dX_s^{\varepsilon_n}\Big|\Big) = 0,\end{aligned}$$

or

$$\int_0^t g(X_s)\,dW_s = \int_0^t g(X_s)\,dX_s, \qquad \text{a.s.}$$

Letting $\delta \to 0$ proves (6.6).

We know

$$X_t^M = \int_0^t 1_{(X_s^M \neq 0)}\,dX_s^M.$$

Since X^M and X have the same law, the same is true if we replace X^M by X. Combining with (6.6) proves (5.2). □

7 Some estimates

Let

$$j^\varepsilon(s) = \begin{cases} 1, & |X_s^\varepsilon| \in [-\varepsilon, \varepsilon] \text{ or } |Y_s^\varepsilon| \in [-2\varepsilon, 2\varepsilon] \text{ or both;} \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$J_t^\varepsilon = \int_0^t j_s^\varepsilon\,ds.$$

Set

$$Z_t^\varepsilon = X_t^\varepsilon - Y_t^\varepsilon,$$

suppose $Z_0^\varepsilon = 0$, and define $\psi_\varepsilon(x,y) = \sigma_\varepsilon(x) - \sigma_{2\varepsilon}(y)$. Then

$$dZ_t^\varepsilon = \psi_\varepsilon(X_t^\varepsilon, Y_t^\varepsilon)\, d\widetilde{W}_t.$$

Let

$$\begin{aligned} S_1 &= \inf\{t : |Z_t^\varepsilon| \ge 6\varepsilon\}, \\ T_i &= \inf\{t \ge S_i : |Z_t^\varepsilon| \notin [4\varepsilon, b]\}, \\ S_{i+1} &= \inf\{t \ge T_i : |Z_t^\varepsilon| \ge 6\varepsilon\}, \qquad \text{and} \\ U_b &= \inf\{t : |Z_t^\varepsilon| = b\}. \end{aligned} \tag{7.1}$$

Proposition 7.1. *For each n,*

$$\mathbb{P}(S_n < U_b) \le \Big(1 - \frac{2\varepsilon}{b}\Big)^n.$$

Proof. Since X^ε is a recurrent diffusion, $\int_0^t 1_{[-\varepsilon,\varepsilon]}(X_s^\varepsilon)\, ds$ tends to infinity a.s. as $t \to \infty$. When $x \in [-\varepsilon, \varepsilon]$, then $|\psi_\varepsilon(x,y)| \ge c\varepsilon$, and we conclude that $\langle Z^\varepsilon \rangle_t \to \infty$ as $t \to \infty$.

Let $\{\mathcal{F}_t\}$ be the filtration generated by $\widetilde{W}$. $Z_{t+S_n}^\varepsilon - Z_{S_n}^\varepsilon$ is a martingale started at 0 with respect to the regular conditional probability for the law of $(X_{t+S_n}^\varepsilon, Y_{t+S_n}^\varepsilon)$ given $\mathcal{F}_{S_n}$. The conditional probability that it hits 4ε before b if $Z_{S_n}^\varepsilon = 6\varepsilon$ is the same as the conditional probability it hits -4ε before $-b$ if $Z_{S_n}^\varepsilon = -6\varepsilon$ and is equal to

$$\frac{b - 6\varepsilon}{b - 4\varepsilon} \le 1 - \frac{2\varepsilon}{b}.$$

Since this is independent of ω, we have

$$\mathbb{P}\Big(|Z_{t+S_n}^\varepsilon - Z_{S_n}^\varepsilon| \text{ hits } 4\varepsilon \text{ before hitting } b \mid \mathcal{F}_{S_n}\Big) \le 1 - \frac{2\varepsilon}{b}.$$

Let $V_n = \inf\{t > S_n : |Z_t^\varepsilon| = b\}$. Then

$$\begin{aligned} \mathbb{P}(S_{n+1} < U_b) &\le \mathbb{P}(S_n < U_b, T_{n+1} < V_n) \\ &= \mathbb{E}\left[\mathbb{P}(T_{n+1} < V_n \mid \mathcal{F}_{S_n}); S_n < U_b\right] \\ &\le \Big(1 - \frac{2\varepsilon}{b}\Big)\mathbb{P}(S_n < U_b). \end{aligned}$$

Our result follows by induction. □

Proposition 7.2. *There exists a constant c_1 such that*

$$\mathbb{E}\, J^\varepsilon_{T_n} \leq c_1 n\varepsilon$$

for each n.

Proof. For t between times S_n and T_n we know that $|Z^\varepsilon_t|$ lies between 4ε and b. Then at least one of $X^\varepsilon_t \notin [-\varepsilon, \varepsilon]$ and $Y^\varepsilon_t \notin [-2\varepsilon, 2\varepsilon]$ holds. If exactly one holds, then $|\psi_\varepsilon(X^\varepsilon_t, Y^\varepsilon_t)| \geq 1 - \sqrt{2\varepsilon/\gamma} \geq 1/2$ if ε is small enough. If both hold, we can only say that $d\langle Z^\varepsilon\rangle_t \geq 0$. In any case,

$$d\langle Z^\varepsilon\rangle_t \geq \tfrac{1}{4}\, dJ^\varepsilon_t$$

for $S_n \leq t \leq T_n$.

Z^ε_t is a martingale, and by Lemma 2.1 and an argument using regular conditional probabilities similar to those we have done earlier,

$$\mathbb{E}\,[J^\varepsilon_{T_n} - J^\varepsilon_{S_n}] \leq 4\mathbb{E}\,[\langle Z^\varepsilon\rangle_{T_n} - \langle Z^\varepsilon\rangle_{S_n}] \leq 4(b - 6\varepsilon)(2\varepsilon) = c\varepsilon. \tag{7.2}$$

Between times T_n and S_{n+1} it is possible that $\psi_\varepsilon(X^\varepsilon_t, Y^\varepsilon_t)$ can be 0 or it can be larger than $c\sqrt{\varepsilon/\gamma}$. However if either $X^\varepsilon_t \in [-\varepsilon, \varepsilon]$ or $Y^\varepsilon_t \in [-2\varepsilon, 2\varepsilon]$, then $\psi_\varepsilon(X^\varepsilon_t, Y^\varepsilon_t) \geq c\sqrt{\varepsilon/\gamma}$. Thus

$$d\langle Z^\varepsilon\rangle_t \geq c\varepsilon\, dJ^\varepsilon_t$$

for $T_n \leq t \leq S_{n+1}$. By Lemma 2.1

$$\mathbb{E}\,[J^\varepsilon_{S_{n+1}} - J^\varepsilon_{T_n}] \leq c\varepsilon^{-1}\mathbb{E}\,[\langle Z^\varepsilon\rangle_{S_{n+1}} - \langle Z^\varepsilon\rangle_{T_n}] \leq c\varepsilon^{-1}(2\varepsilon)(10\varepsilon) = c\varepsilon. \tag{7.3}$$

Summing each of (7.2) and (7.3) over j from 1 to n and combining yields the proposition. □

Proposition 7.3. *Let $K > 0$ and $\eta > 0$. There exists R depending on K and η such that*

$$\mathbb{P}(J^\varepsilon_{\tau_{[-R,R]}(X^\varepsilon)} < K) \leq \eta, \qquad \varepsilon \leq 1/2.$$

Proof. Fix $\varepsilon \le 1/2$. We will see that our estimates are independent of ε. Note

$$J_t^\varepsilon \ge H_t = \int_0^t 1_{[-\varepsilon,\varepsilon]}(X_s^\varepsilon)\,ds.$$

Therefore to prove the proposition it is enough to prove that

$$\mathbb{P}_\varepsilon^0(H_{\tau_{[-R,R]}(X^\varepsilon)} < K) \le \eta$$

if R is large enough.

Let $I = [-1,1]$. We have

$$\mathbb{E}_\varepsilon^{\,0} H_{\tau_I(X^\varepsilon)} \ge \int_{-1}^1 g_I(0,y)\frac{\gamma}{\varepsilon} 1_{[-\varepsilon,\varepsilon]}(y)\,dy \ge c_1.$$

On the other hand, for any $x \in I$,

$$\mathbb{E}_0^{\,x} H_{\tau_I(X^\varepsilon)} = \int_I g_I(x,y)\frac{\gamma}{\varepsilon}1_{[-\varepsilon,\varepsilon]}(y)\,dy \le c_2.$$

Combining this with

$$\mathbb{E}_\varepsilon^{\,0}[H_{\tau_I(X^\varepsilon)} - H_t \mid \mathcal{F}_t] \le \mathbb{E}_\varepsilon^{\,X_t^\varepsilon} H_{\tau_I(X^\varepsilon)}$$

and Theorem I.6.10 of [4] (with $B = c_2$ there), we see that

$$\mathbb{E}\, H^2_{\tau_I(X^\varepsilon)} \le c_3.$$

Let $\alpha_0 = 0$, $\beta_i = \inf\{t > \alpha_i : |X_t^\varepsilon| = 1\}$ and $\alpha_{i+1} = \inf\{t > \beta_i : X_t^\varepsilon = 0\}$. Since X_t^ε is a recurrent diffusion, each α_i is finite a.s. and $\beta_i \to \infty$ as $i \to \infty$. Let $V_i = H_{\beta_i} - H_{\alpha_i}$. By the strong Markov property, under $\mathbb{P}_\varepsilon^0$ the V_i are i.i.d. random variables with mean larger than c_1 and variance bounded by c_4, where c_1 and c_4 do not depend on ε as long as $\varepsilon < 1/2$. Then

$$\begin{aligned}
\mathbb{P}_\varepsilon^0\Big(\sum_{i=1}^k V_i \le c_1 k/2\Big) &\le \mathbb{P}_\varepsilon^0\Big(\sum_{i=1}^k (V_i - \mathbb{E}\,V_i) \ge c_1 k/2\Big)\\
&\le \frac{\mathrm{Var}\,(\sum_{i=1}^k V_i)}{(c_1 k/2)^2}\\
&\le 4c_4/c_1^2 k.
\end{aligned}$$

Taking k large enough, we see that

$$\mathbb{P}^0_\varepsilon\Big(\sum_{i=1}^k V_i \le K\Big) \le \eta/2.$$

Using the fact that X^ε_t is a martingale, starting at 1, the probability of hitting R before hitting 0 is $1/R$. Using the strong Markov property, the probability of $|X|$ having no more than k downcrossings of $[0,1]$ before exiting $[-R,R]$ is bounded by

$$1-\Big(1-\frac{1}{R}\Big)^k.$$

If we choose R large enough, this last quantity will be less than $\eta/2$. Thus, except for an event of probability at most η, X^ε_t will exit $[-1,1]$ and return to 0 at least k times before exiting $[-R,R]$ and the total amount of time spent in $[-\varepsilon,\varepsilon]$ before exiting $[-R,R]$ will be at least K. □

Proposition 7.4. *Let $\eta>0, R>0$, and $I=[-R,R]$. There exists t_0 depending on R and η such that*

$$\mathbb{P}^0_\varepsilon(\tau_I(X^\varepsilon)>t_0)\le\eta, \qquad \varepsilon\le 1/2.$$

Proof. If $\varepsilon\le 1$,

$$\mathbb{E}^{\,0}_\varepsilon \tau_R(X^\varepsilon)=\int_I g_I(x,y)\, m_\varepsilon(dy).$$

A calculation shows this is bounded by cR^2+cR, where c does not depend on ε or R. Applying Chebyshev's inequality,

$$\mathbb{P}^0_\varepsilon(\tau_I(X^\varepsilon)>t_0)\le\frac{\mathbb{E}^{\,0}_\varepsilon\tau_I(X^\varepsilon)}{t_0},$$

which is bounded by η if $t_0\ge c(R^2+R)/\eta$. □

8 Pathwise uniqueness fails

We continue the notation of Section 7. The strategy of proving that pathwise uniqueness does not hold owes a great deal to [2].

Theorem 8.1. *There exist three processes X, Y, and W and a probability measure $\mathbb{P}$ such that W is a Brownian motion under $\mathbb{P}$, X and Y are continuous martingales under $\mathbb{P}$ with speed measure m starting at 0,* (5.2) *holds for X,* (5.2) *holds when X is replaced by Y, and $\mathbb{P}(X_t \neq Y_t$ for some $t > 0) > 0$.*

Proof. Let $(X^\varepsilon, Y^\varepsilon, \widetilde{W})$ be defined as in (6.1) and (6.2) and choose a sequence ε_n decreasing to 0 such that the triple converges weakly on $C[0,N]\times C[0,N]\times C[0,N]$ for each N. By Theorems 6.1 and 6.2, the weak limit, (X,Y,W) is such that X and Y are continuous martingales with speed measure m, W is a Brownian motion, and (5.2) holds for X and also when X is replaced by Y.

Let $b = 1$ and let S_n, T_n, and U_b be defined by (7.1). Let $A_1(\varepsilon, n)$ be the event where $T_n < U_b$. By Proposition 7.1

$$\mathbb{P}(A_1(\varepsilon,n)) = \mathbb{P}(S_n < U_b) \le \Big(1 - \frac{2\varepsilon}{b}\Big)^n.$$

Choose $n \ge \beta/\varepsilon$, where β is large enough so that the right hand side is less than $1/5$ for all ε sufficiently small.

By Proposition 7.2,

$$\mathbb{E}\, J^\varepsilon_{T_n} \le c_1 n\varepsilon = c_1\beta.$$

By Chebyshev's inequality,

$$\mathbb{P}(J^\varepsilon_{T_n} \ge 5c_1\beta) \le \mathbb{P}(J^\varepsilon_{T_n} \ge 5\mathbb{E}\, J^\varepsilon_{T_n}) \le 1/5.$$

Let $A_2(\varepsilon,n)$ be the event where $J^\varepsilon_{T_n} \ge 5c_1\beta$.

Take $K = 10c_1\beta$. By Proposition 7.3, there exists R such that

$$\mathbb{P}(J^\varepsilon_{\tau_{[-R,R]}(X^\varepsilon)} < K) \le 1/5.$$

Let $A_3(\varepsilon, R, K)$ be the event where $J^\varepsilon_{\tau_{[R,R]}(X^\varepsilon)} < K$.

Choose t_0 using Proposition 7.4, so that except for an event of probability $1/5$ we have $\tau_{[-R,R]}(X^\varepsilon) \le t_0$. Let $A_4(\varepsilon, R, t_0)$ be the event where $\tau_{[-R,R]}(X^\varepsilon) \le t_0$.

Let

$$B(\varepsilon) = (A_1(\varepsilon, n) \cup A_2(\varepsilon, n) \cup A_3(\varepsilon, R, K) \cup A_4(\varepsilon, R, t_0))^c.$$

Note $\mathbb{P}(B(\varepsilon)) \geq 1/5$.

Suppose we are on the event $B(\varepsilon)$. We have

$$J^\varepsilon_{T_n} \leq 5c_1\beta < K \leq J^\varepsilon_{\tau_{[-R,R]}(X^\varepsilon)}.$$

We conclude that $T_n < \tau_{[-R,R]}(X^\varepsilon)$. Therefore, on the event $B(\varepsilon)$, we see that T_n has occurred before time t_0. We also know that U_b has occurred before time t_0. Hence, on $B(\varepsilon)$,

$$\mathbb{P}(\sup_{s \leq t_0} |Z^\varepsilon_s| \geq b) \geq 1/5.$$

Since $Z^\varepsilon = X^\varepsilon - Y^\varepsilon$ converges weakly to $X - Y$, then with probability at least $1/5$, we have that $\sup_{s \leq t_0} |Z_s| \geq b/2$. This implies that $X_t \neq Y_t$ for some t, or pathwise uniqueness does not hold. □

We also can conclude that strong existence does not hold. The argument we use is similar to ones given in [8], [10], and [18].

Theorem 8.2. *Let W be a Brownian motion. There does not exist a continuous martingale X starting at 0 with speed measure m such that* (5.2) *holds and such that X is measurable with respect to the filtration of W.*

Proof. Let W be a Brownian motion and suppose there did exist such a process X. Then there is a measurable map $F : C[0, \infty) \to C[0, \infty)$ such that $X = F(W)$.

Suppose Y is any other continuous martingale with speed measure m satisfying (5.2). Then by Theorem 4.1, the law of Y equals the law of X, and by Theorem 5.2, the joint law of (Y, W) is equal to the joint law of (X, W). Therefore Y also satisfies $Y = F(W)$, and we get pathwise uniqueness since $X = F(W) = Y$. However, we know pathwise uniqueness does not hold. We conclude that no such X can exist, that is, strong existence does not hold. □

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