Correction to "Meyers inequality and strong stability for stable-like operators" by Bass and Ren

There is an error in the published version of this paper. The scaling argument appealed to in the line following (4.11) does not work. This was pointed out to us by Prof. Tadele Mengesha. What is needed is to restrict attention to R less than $4\sqrt{d}$. That argument is given here.

A. Proof of Proposition 4.3

[*Equation and proposition numbers refer to the original paper.*]

Proof. Set $x_0 = 0$ for now. From (4.1) we know that

$$\begin{aligned} \|\Gamma u\|^2_{L^2(B_{R/2})} &\le c\int_{\mathbb{R}^d} (u(x)-u_R)^2\psi_R(x)\,dx + \int_{B_R} |h(x)(u(x)-u_R)|\,dx \\ &\le cR^{-\alpha}\int_{B_R} (u(x)-u_R)^2\,dx + c\int_{B_R^c} u(x)^2\psi_R(x)\,dx \\ &\quad + c\int_{B_R^c} u_R^2\psi_R(x)\,dx + \int_{B_R} |h(x)(u(x)-u_R)|\,dx \\ &= J_1 + J_2 + J_3 + J_4; \end{aligned} \tag{A.1}$$

recall

$$\psi_R(x) = R^{-\alpha} \wedge \frac{R^d}{|x-x_0|^{d+\alpha}}.$$

We proceed to bound J_1, J_2, J_3, and J_4.

Using Proposition 4.2, we have

$$J_1 \le cR^{-\alpha_1}\Big(\int_{B_R} \Gamma u(x)^{q_1}\,dx\Big)^{\frac{2}{q_1}} \tag{A.2}$$

for $q_1 \in (1,2)$.

Let M be the Hardy-Littlewood maximal operator:

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B(x,r)} |f(z)|\,dz.$$

If $y \in B_R$ and $k \geq 0$, then

$$\frac{1}{|B_{2^kR}|} \int_{B_{2^kR}} u(x)^2\,dx \leq \frac{1}{|B_{2^kR}|} \int_{B(y,(2^k+1)R)} u(x)^2\,dx \leq cM(u^2)(y).$$

Similarly, if $x \in B_R$,

$$|u_R| \leq \frac{1}{|B_R|} \int_{B_R} |u(z)|\,dz \leq cMu(x).$$

For J_2, we then have

$$\begin{aligned}
c\int_{B_R^c} u(x)^2 \psi_R(x)\,dx &= c\sum_{n=0}^{\infty} \int_{\{B_{2^{n+1}R} - B_{2^nR}\}} u(x)^2 \frac{R^d}{|x|^{d+\alpha}} dx \\
&\leq c\sum_{n=0}^{\infty} \frac{(2^{n+1}R)^d}{|B_{2^{n+1}R}|} \int_{B_{2^{n+1}R}} u(x)^2\,dx\, \frac{R^d}{(2^nR)^{d+\alpha}} \\
&\leq c\sum_{n=0}^{\infty} M(u^2)(y) 2^{nd} \frac{R^{2d}}{2^{nd}2^{n\alpha}R^{d+\alpha}} \\
&= cM(u^2)(y)R^{d-\alpha} \sum_{n=0}^{\infty} \frac{1}{2^{n\alpha}} \\
&= cM(u^2)(y)R^{d-\alpha},
\end{aligned}$$

as long as $y \in B_R$.

For J_3 we have

$$\begin{aligned}
c\int_{B_R^c} u_R^2 \psi_R(x)\,dx &= cu_R^2 \int_{B_R^c} \frac{R^d}{|x|^{d+\alpha}}\,dx \\
&= cR^{d-\alpha} u_R^2 \leq cR^{d-\alpha} M(u^2)(y)
\end{aligned}$$

if $y \in B_R$.

Since $|B(x,s)|^{-1}\int_{B(x,s)} u(y)\,dy$ converges to $u(x)$ as $s\to 0$ for almost every x and is bounded by $Mu(x)$, we have $|u(x)|\le Mu(x)$ a.e. Thus, with $x\in B_R$,

$$J_4 = c\int_{B_R} |h(x)(u(x)-u_R)|\,dx \le c\int_{B_R} |h(x)u(x)|\,dx + c\int_{B_R} |h(x)Mu(x)|\,dx \le c\int_{B_R} |h(x)|Mu(x)\,dx.$$

Combining our bounds for J_1, J_2, J_3, and J_4, if $y\in B_R$,

$$\|\Gamma u\|^2_{L^2(B_{R/2})} \le cR^{-\alpha_1}\|\Gamma u\|^2_{L^{q_1}(B_R)} + cR^{d-\alpha}M(u^2)(y) + c\int_{B_R} |h(x)|Mu(x)\,dx. \tag{A.3}$$

Integrating both sides of (A.3) over $y\in B_R$ and dividing by $|B_R|$, we conclude that

$$\int_{B_{R/2}} \Gamma u(x)^2\,dx \le cR^{-\alpha_1}\Big(\int_{B_R}\Gamma u(x)^{q_1}\,dx\Big)^{\frac{2}{q_1}} + cR^{d-\alpha}\int_{B_R} M(u^2)(x)\,dx + c\int_{B_R}|h(x)|Mu(x)\,dx. \tag{A.4}$$

Let

$$g(x) = \Gamma u(x)^{q_1}$$

and

$$f(x) = \Big(M(u^2)(x) + |h(x)|Mu(x)\Big)^{\frac{q_1}{2}}.$$

Set $R_0 = 4\sqrt{d}$ and suppose from now on that $R<R_0$. Recall that we assume $d\ge\alpha$ (see the second paragraph of Section 2). Noting that $R^{d-\alpha}$ is then bounded by $(4\sqrt{d})^{d-\alpha}$ and recalling that $\alpha_1 = (2-q_1)d/q_1$, we can rewrite (A.4) as

$$\frac{1}{|B(x_0,R)|}\int_{B(x_0,R/2)} g^{\frac{2}{q_1}}(x)\,dx \le c\Big(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} g(x)\,dx\Big)^{\frac{2}{q_1}} + c\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} f^{\frac{2}{q_1}}(x)\,dx \tag{A.5}$$

if $R < R_0$. By a translation argument, (A.5) holds for all $x_0 \in \mathbb{R}^d$.

We now apply the reverse Hölder inequality (see Theorem 4.1 in [12]). Thus there exists $\varepsilon > 0$ and $c_1 > 0$ such that if $R < R_0$, then $g(x) \in L^t(B(x_0, R/2))$ for all $t \in [\frac{2}{q_1}, \frac{2}{q_1} + \varepsilon)$ and

$$\Big(\frac{1}{|B(x_0,R/2)|}\int_{B(x_0,R/2)} g^t(x)\,dx\Big)^{\frac{1}{t}} \le c\Big(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} g^{\frac{2}{q_1}}(x)\,dx\Big)^{\frac{q_1}{2}} + c\Big(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} f^t(x)\,dx\Big)^{\frac{1}{t}}.$$

This leads to

$$\begin{aligned}\Big(\frac{1}{|B(x_0,R/2)|}&\int_{B(x_0,R/2)} \Gamma u(x)^{q_1 t}\,dx\Big)^{\frac{1}{t}}\\ &\le c\Big(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} \Gamma u(x)^2\,dx\Big)^{\frac{q_1}{2}}\\ &\quad + c\Big(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} (M(u^2))^{tq_1/2}(x)\,dx\Big)^{\frac{1}{t}}\\ &\quad + c\Big(\frac{1}{|B(x_0,R)|}\int (|h|Mu)^{tq_1/2}\,dx\Big)^{1/t}.\end{aligned}$$

Choose $t \in (2/q_1, 2/q_1 + \varepsilon)$ so that $q_1 t < 4d/(d-\alpha)$ and set $p = q_1 t$.

Now set $R = 2\sqrt{d}$ for the remainder of the proof. Taking q_1^{th} roots and using the inequality $(a+b)^{1/q_1} \le a^{1/q_1} + b^{1/q_1}$,

$$\|\Gamma u\|_{L^p(B(x_0,R/2))} \le c\|\Gamma u\|_{L^2(B(x_0,R))} + c\|M(u^2)\|^{1/2}_{L^{p/2}(B(x_0,R))} + c\|h(Mu)\|^{1/2}_{L^{p/2}(B(x_0,R))}.$$

[*The remainder of the proof is as in the original paper.*]