Meyers inequality and strong stability for stable-like operators

Richard F. Bass and Hua Ren*

September 1, 2012

Abstract

Let $\alpha \in (0,2)$, let

$$\mathcal{E}(u,u) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y)-u(x))^2 \frac{A(x,y)}{|x-y|^{d+\alpha}}\, dy\, dx$$

be the Dirichlet form for a stable-like operator, let

$$\Gamma u(x) = \int_{\mathbb{R}^d} (u(y)-u(x))^2 \frac{A(x,y)}{|x-y|^{d+\alpha}}\, dy,$$

let L be the associated infinitesimal generator, and suppose $A(x,y)$ is jointly measurable, symmetric, bounded, and bounded below by a positive constant. We prove that if u is the weak solution to $Lu = h$, then $\Gamma u \in L^p$ for some $p > 2$. This is the analogue of an inequality of Meyers for solutions to divergence form elliptic equations. As an application, we prove strong stability results for stable-like operators. If A is perturbed slightly, we give explicit bounds on how much the semigroup and fundamental solution are perturbed.

Subject Classification: Primary 45K05; Secondary 35B65

*Research of both authors partially supported by NSF grant DMS-0901505.

1 Introduction

Nowadays many researchers who use mathematical models consider situations where discontinuities can occur. In analysis terms, this means they need to look at integro-differential operators as well as differential operators. Integro-differential operators are not nearly as well understood as their differential counterparts, and to study them it makes sense to first look at the extreme case, that of purely integral operators.

In this paper we focus on a reasonably large class of such integral operators, the stable-like operators. These are operators that bear the same relationship to the fractional Laplacian as divergence form operators do to the Laplacian.

To describe our results, let us first recall some facts about divergence form operators. These have the form

$$\mathcal{L}_d f(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(\cdot)\frac{\partial f}{\partial x_j}(\cdot)\Big)(x).$$

These have been studied even when the a_{ij} are only bounded and measurable, and to make sense of the operator in this case, one looks at the corresponding Dirichlet form:

$$\mathcal{E}_d(f,f) = \int_{\mathbb{R}^d} \sum_{i,j=1}^{d} \frac{\partial f}{\partial x_i}(x)\frac{\partial f}{\partial x_j}(x)\,dx.$$

One says that u is a weak solution of $\mathcal{L}_d u = h$ if $\mathcal{E}_d(u,v) = -(h,v)$ for all v in a suitably large class, where $(h,v) = \int_{\mathbb{R}^d} h(x)v(x)\,dx$.

An inequality of Meyers ([25]) says that if the a_{ij} are uniformly elliptic and u is a weak solution to $\mathcal{L}_d u = h$, then not only is ∇u locally in L^2 but it is locally in L^p for some $p > 2$.

The Meyers inequality has many applications. One is to the stability of solutions to $\mathcal{L}_d u = h$. Suppose one perturbs the coefficients a_{ij} slightly. How does this affect the associated semigroup? What about the fundamental solution associated with the operator $\mathcal{L}_d$? These are natural questions since the coefficients a_{ij} might themselves be only estimated or approximated. In [18] these issues were resolved, with an explicit bound on how large the difference between the semigroups and solutions associated with two operators $\mathcal{L}_d$ and $\widetilde{\mathcal{L}}_d$ can be in terms of the difference of the coefficients a_{ij} and $\widetilde{a}_{ij}$.

Our purpose in this paper is to examine the analogues of these results for stable-like processes. The operator we consider is

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} (f(y) - f(x)) \frac{A(x,y)}{|x-y|^{d+\alpha}}\, dy,$$

where $\alpha \in (0,2)$ and $A(x,y)$ is bounded, symmetric, jointly measurable, and bounded below. As in the case for divergence form operators, it is useful to look at the associated Dirichlet form

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(y) - f(x))^2 \frac{A(x,y)}{|x-y|^{d+\alpha}}\, dy\, dx.$$

The bulk of this paper is devoted to proving a Meyers inequality for weak solutions to $\mathcal{L}u = h$ when h is in L^2. Define

$$\Gamma u(x) = \Big(\int_{\mathbb{R}^d} \frac{(u(y) - u(x))^2}{|x-y|^{d+\alpha}}\, dy \Big)^{\frac{1}{2}}. \tag{1.1}$$

Our main result is that there exists $p > 2$ such that the L^p norm of Γu is bounded in terms of the L^2 norms of u and h; see Theorem 4.4.

Once one has the Meyers inequality for $\mathcal{E}$, strong stability results can be proved along the lines of [18]. Suppose $\widetilde{\mathcal{E}}$ is defined in terms of $\widetilde{A}(x,y)$ analogously to (1.1). We obtain explicit bounds on the L^p norm of $P_t f - \widetilde{P}_t f$ and on the L^∞ norm of $p(t,x,y) - \widetilde{p}(t,x,y)$ in terms of

$$G(x) = \sup_{y \in R^d} |A(x,y) - \widetilde{A}(x,y)|,$$

where P_t and $p(t,\cdot,\cdot)$ are the semigroup and fundamental solution associated with $\mathcal{L}$ and $\widetilde{P}_t$ and $\widetilde{p}(t,\cdot,\cdot)$ are defined similarly. See Theorems 5.1, 5.2, and 5.3.

Our proof of the Meyers inequality begins by first proving a Caccioppoli inequality. However there are considerable differences between the stable-like case and the divergence form case. For example, as one would expect, our Caccioppoli inequality is not a local one; the integral of $|\Gamma u|^2$ on a ball depends on values of u far outside the ball. This makes proving the Meyers inequality considerably more difficult and requires the introduction of some new ideas.

For other papers on stable-like operators and on closely related operators, see [2] – [11], [13] – [17], [21], [22], [24], and [27].

2 Preliminaries

We use the letter c with or without subscripts to denote a finite positive constant whose exact value is unimportant and which can vary from place to place. We use $B(x,r)$ for the open ball in $\mathbb{R}^d$ with center x and radius r. When the center is clear from the context, we will also write B_r. The Lebesgue measure of $B(x,r)$ will be denoted $|B(x,r)|$. We write (u,v) for $\int_{\mathbb{R}^d} u(x)v(x)\,dx$.

Let $\alpha \in (0,2)$ and suppose the dimension d is greater than α. We let $A(x,y)$ be a jointly measurable symmetric function on $\mathbb{R}^d \times \mathbb{R}^d$ and suppose there exists $\Lambda > 0$ such that

$$\Lambda^{-1} \le A(x,y) \le \Lambda, \qquad x,y \in \mathbb{R}^d.$$

We define the Dirichlet form $\mathcal{E}$ with domain $\mathcal{D}(\mathcal{E}) = \mathcal{F}$ by

$$\begin{aligned}\mathcal{E}(u,v) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(y)-u(x))(v(y)-v(x))\frac{A(x,y)}{|x-y|^{d+\alpha}}\,dy\,dx,\\ \mathcal{F} &= \{u \in L^2(\mathbb{R}^d) : \mathcal{E}(u,u) < \infty\}.\end{aligned}$$

Observe that $\mathcal{F} = W^{\frac{\alpha}{2},2}(\mathbb{R}^d)$, the fractional Sobolev space of order $\alpha/2$, defined by

$$W^{\frac{\alpha}{2},2}(\mathbb{R}^d) = \Big\{u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,dy\,dx < \infty\Big\}.$$

See [1] for more details. It is well known that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d)$. The strong Markov symmetric process X associated with $(\mathcal{E},\mathcal{F})$ is called a stable-like process. Let $\{P_t\}_{t\ge 0}$ be the semigroup corresponding to $(\mathcal{E},\mathcal{F})$.

For $u \in \mathcal{F}$ define

$$\Gamma u(x) = \Big(\int_{\mathbb{R}^d}\frac{(u(y)-u(x))^2}{|x-y|^{d+\alpha}}\,dy\Big)^{\frac{1}{2}}. \tag{2.1}$$

Since $\int |\Gamma u(x)|^2\,dx = \mathcal{E}(u,u) < \infty$, then $\Gamma u \in L^2$, and in particular $\Gamma u(x)$ exists for almost every x.

If $\mathcal{L}$ is the infinitesimal generator corresponding to $\mathcal{E}$ (see [23]), there are a number of known results that follow from the spectral theorem. We collect these for the convenience of the reader. Let $\{E_\lambda\}$, $\lambda \geq 0$, be the spectral representation of $-\mathcal{L}$. For $f \in \mathcal{F}$, we have

$$\mathcal{E}(f,f) = \int_0^\infty \lambda \, d(E_\lambda f, E_\lambda f);$$

see [23].

Lemma 2.1. *For $t > 0$, $f \in L^2(D)$, we have*

$$\mathcal{E}(P_t f, P_t f) \leq c t^{-1} \|f\|_2^2.$$

Proof. This follows from

$$\begin{aligned}
\mathcal{E}(P_t f, P_t f) &= \int_0^\infty \lambda e^{-2\lambda t} \, d(E_\lambda f, E_\lambda f) \\
&\leq c t^{-1} \int_0^\infty d(E_\lambda f, E_\lambda f) = c t^{-1} \|f\|_2^2,
\end{aligned}$$

since $\lambda e^{-2\lambda t} \leq c t^{-1}$ for all $\lambda \geq 0$. □

Lemma 2.2. *If $g \in L^2$, then $P_t g$ is in $\mathcal{D}(\mathcal{L})$, the domain of $\mathcal{L}$.*

Proof. By the spectral representation of $-\mathcal{L}$, we have

$$\frac{P_h(P_t g) - P_t g}{h} = \frac{P_{t+h} g - P_t g}{h} = \int_0^\infty \frac{e^{-\lambda(t+h)} - e^{-\lambda t}}{h} \, dE_\lambda g.$$

Let $H = -\int_0^\infty \lambda e^{-\lambda t} \, dE_\lambda g$. Note $\|H\|_{L^2}$ is finite because $\lambda^2 e^{-2\lambda t}$ is bounded. Then

$$\left\| \frac{P_h(P_t g) - P_t g}{h} - H \right\|_{L^2}^2 = \int_0^\infty \left[\frac{e^{-\lambda(t+h)} - e^{-\lambda t}}{h} + \lambda e^{-\lambda t} \right]^2 d(E_\lambda g, E_\lambda g),$$

which tends to 0 as $h \to 0$ by dominated convergence. Therefore $P_t g \in \mathcal{D}(\mathcal{L})$ and $\mathcal{L}(P_t g) = H$. □

Lemma 2.3. *If* $f, g \in \mathcal{F}$, *then*

$$\frac{d}{dt}(P_t f, g) = -\mathcal{E}(P_t f, g).$$

Proof. For any $g \in \mathcal{F}$, we have

$$(P_t f, g) = \int_0^\infty e^{-\lambda t}\, d(E_\lambda f, g),$$

and so

$$\frac{d}{dt}(P_t f, g) = -\int_0^\infty \lambda e^{-\lambda t}\, d(E_\lambda f, g).$$

On the other hand,

$$\mathcal{E}(P_t f, g) = \int_0^\infty \lambda\, d(E_\lambda P_t f, g) = \int_0^\infty \lambda e^{-\lambda t}\, d(E_\lambda f, g),$$

which proves the lemma. □

Lemma 2.4. *If* $f \in \mathcal{F}$, *then*

$$\mathcal{E}(P_t f, P_t f) \leq \mathcal{E}(f, f). \tag{2.2}$$

Proof. We prove this by writing

$$\int_0^\infty \lambda e^{-2\lambda t}\, d(E_\lambda f, E_\lambda f) \leq \int_0^\infty \lambda\, d(E_\lambda f, E_\lambda f),$$

which translates to (2.2). □

3 Caccioppoli inequality

In this section, we will derive a Caccioppoli inequality for the weak solution of the equation

$$\mathcal{L}u(x) = h(x), \qquad x \in \mathbb{R}^d, \tag{3.1}$$

where $h \in L^2(\mathbb{R}^d)$. A function $u \in W^{\frac{\alpha}{2},2}(\mathbb{R}^d)$ is called a weak solution of (3.1) if

$$\mathcal{E}(u, v) = -(h, v) \text{ for all } v \in W^{\frac{\alpha}{2},2}(\mathbb{R}^d), \tag{3.2}$$

where $(h, v) = \int h(x)v(x)\, dx$.

Theorem 3.1. *Let $x_0 \in \mathbb{R}^d$. Suppose $u(x)$ satisfies* (3.1). *There exists a constant c_1 depending only on Λ and d such that*

$$\int_{B_{R/2}} \int_{\mathbb{R}^d} (u(y)-u(x))^2 \frac{A(x,y)}{|x-y|^{d+\alpha}}\, dy\, dx \\ \le c_1 \int_{\mathbb{R}^d} u^2(y)\psi(y)\, dy + \int_{B_R} |h(y)u(y)|\, dy, \tag{3.3}$$

where

$$\psi(x) = R^{-\alpha} \wedge \frac{R^d}{|x-x_0|^{d+\alpha}}.$$

Proof. We define a cutoff function $\varphi(x) : \mathbb{R}^d \to [0,1]$ such that $\varphi = 1$ on $B_{R/2}$, $\varphi = 0$ on B_R^c, and

$$|\varphi(x) - \varphi(y)| \le c\frac{|x-y|}{R}.$$

For example, we can take

$$\varphi(x) = 1 - \Big(\frac{\mathrm{dist}\,(x, B(x_0, R/2))}{R/2} \wedge 1 \Big).$$

Let $v(x) = \varphi^2(x)u(x)$. Since $|v| \le |u|$ and $u \in L^2$, then $v \in L^2$. Since

$$v(y) - v(x) = (u(y)-u(x))\varphi^2(y) + u(x)(\varphi^2(y) - \varphi^2(x)),$$

then

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(v(y)-v(x))^2}{|x-y|^{d+\alpha}}\, dy\, dx \le 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(u(y)-u(x))^2\varphi^4(y)}{|x-y|^{d+\alpha}}\, dy\, dx \\ + 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{u^2(x)(\varphi^2(y)-\varphi^2(x))^2}{|x-y|^{d+\alpha}}\, dy\, dx.$$

The first term on the right hand side is finite because $\varphi \le 1$ and $u \in \mathcal{F}$. The second term is bounded by

$$c\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{u^2(x)(1 \wedge |y-x|^2/R^2)}{|x-y|^{d+\alpha}}\, dy\, dx \le c\int_{\mathbb{R}^d} u^2(x)\, dx,$$

which is finite since $u \in L^2$. Therefore $v \in \mathcal{F}$.

We write

$$
\begin{aligned}
-(h,v) &= \mathcal{E}(u,v) \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(y)-u(x))(\varphi^2(y)u(y)-\varphi^2(x)u(x)) \frac{A(x,y)}{|x-y|^{d+\alpha}}\,dy\,dx \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(y)-u(x))^2\varphi^2(x) \frac{A(x,y)}{|x-y|^{d+\alpha}}\,dy\,dx \\
&\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} [(u(y)-u(x))(\varphi(y)-\varphi(x))(\varphi(y)+\varphi(x))u(y)] \\
&\qquad \times \frac{A(x,y)}{|x-y|^{d+\alpha}}\,dy\,dx \\
&= I_1 - I_2.
\end{aligned}
$$

Then

$$
\begin{aligned}
I_1 &= I_2 - \int_{\mathbb{R}^d} h(y)\varphi^2(y)u(y)\,dy \\
&\le I_2 + \int_{B_R} |h(y)u(y)|\,dy. \qquad (3.4)
\end{aligned}
$$

Using the inequality $ab \le \frac{1}{8}a^2 + 2b^2$, symmetry, and the fact that $0 \le \varphi(x) \le 1$, we have

$$
\begin{aligned}
I_2 &\le \tfrac{1}{8}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(y)-u(x))^2(\varphi(y)+\varphi(x))^2 \frac{A(x,y)}{|x-y|^{d+\alpha}}\,dy\,dx \\
&\quad + 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\varphi(y)-\varphi(x))^2u^2(y) \frac{A(x,y)}{|x-y|^{d+\alpha}}\,dy\,dx \\
&\le \tfrac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(y)-u(x))^2\varphi^2(x) \frac{A(x,y)}{|x-y|^{d+\alpha}}\,dy\,dx \\
&\quad + 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\varphi(y)-\varphi(x))^2u^2(y) \frac{A(x,y)}{|x-y|^{d+\alpha}}\,dy\,dx \\
&= \tfrac{1}{2}I_1 + 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\varphi(y)-\varphi(x))^2u^2(y) \frac{A(x,y)}{|x-y|^{d+\alpha}}\,dy\,dx.
\end{aligned}
$$

Therefore

$$\tfrac{1}{2}I_1 \le 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\varphi(y)-\varphi(x))^2u^2(y)\frac{A(x,y)}{|x-y|^{d+\alpha}}\,dy\,dx + \int_{B_R}|h(y)u(y)|\,dy. \tag{3.5}$$

Next, using $|\varphi(y)-\varphi(x)| \le c(1\wedge|x-y|/R)$, some calculus shows that

$$\int_{\mathbb{R}^d}(\varphi(y)-\varphi(x))^2\frac{A(x,y)}{|x-y|^{d+\alpha}}\,dx \le cR^{-\alpha}, \qquad y\in\mathbb{R}^d. \tag{3.6}$$

If $y \notin B_{2R}$, then

$$\int_{\mathbb{R}^d}(\varphi(y)-\varphi(x))^2\frac{A(x,y)}{|x-y|^{d+\alpha}}\,dx \le c\int_{B_R}\frac{dx}{|y-x_0|^{d+\alpha}} = c\frac{R^d}{|y-x_0|^{d+\alpha}}.$$

Hence the first term on the right hand side of (3.5) is bounded by

$$\int u(y)^2\psi(y)\,dy. \tag{3.7}$$

Combining (3.5) and (3.7) with the fact that

$$I_1 \ge \int_{B_{R/2}}\int_{\mathbb{R}^d}(u(y)-u(x))^2\frac{A(x,y)}{|x-y|^{d+\alpha}}\,dy\,dx$$

completes the proof. □

4 Meyers inequality

Let $h\in L^2$. We consider the weak solution $u(x)$ of (3.1):

$$\mathcal{E}(u,v) = -(h,v)$$

for all $v\in W^{\frac{\alpha}{2},2}(\mathbb{R}^d)$. We will show that Γu is in L^p for some $p>2$. We suppose throughout this section that $d>\alpha$. This will always be the case if $d\ge 2$.

Let

$$u_R = \frac{1}{|B_R|}\int_{B_R} u(y)\,dy.$$

Using Theorem 3.1 with u replaced by $u - u_R$, we have

$$\|\Gamma u\|^2_{L^2(B_{R/2})} \le c\int_{\mathbb{R}^d}(u(x)-u_R)^2\psi(x)\,dx + \int_{B_R}|h(x)(u(x)-u_R)|\,dx. \tag{4.1}$$

Lemma 4.1. *Suppose $u \in W^{\frac{\alpha}{2},q}(B_R)$, $1 < q \le 2$. Suppose $x_0 \in \mathbb{R}^d$ and $R > 0$. Let $p = 2dq/(2d - q\alpha)$. Then $u \in L^p(B_R)$ and there exists a constant c_1 depending only on d and q such that*

$$\|u - u_R\|_{L^p(B_R)} \le c_1\Big[\int_{B_R}\int_{B_R}\frac{(u(y)-u(x))^q}{|x-y|^{d+\frac{\alpha}{2}q}}\,dy\ dx\Big]^{\frac{1}{q}}. \tag{4.2}$$

Proof. We do the case $R = 1$. The case for general R follows by a scaling argument.

By the Sobolev-Besov embeding theorem (see Theorem 7.57 in [1] or Section 2.3.3 in [19]), we know

$$\begin{aligned}\|u - u_R\|_{L^p(B_1)} &\le c\|u - u_R\|_{W^{\frac{\alpha}{2},q}(B_1)} \\ &= c\Big\{\|u - u_R\|_{L^q(B_1)} + \Big[\int_{B_1}\int_{B_1}\frac{(u(y)-u(x))^q}{|x-y|^{d+\frac{\alpha}{2}q}}\,dy\,dx\Big]^{\frac{1}{q}}.\Big\}\end{aligned} \tag{4.3}$$

On the other hand, the fractional Poincaré inequality for $u \in W^{\frac{\alpha}{2},q}(B_1)$ (see equation (4.2) in [26]) tells us

$$\|u - u_R\|_{L^q(B_1)} \le c\Big[\int_{B_1}\int_{B_1}\frac{(u(y)-u(x))^q}{|x-y|^{d+\frac{\alpha}{2}q}}\,dy\,dx\Big]^{\frac{1}{q}}. \tag{4.4}$$

Combining (4.3) and (4.4) proves the lemma. □

Proposition 4.2. *There exists $q_1 \in (1,2)$ and a constant c_1 depending on d and q_1 such that if $x_0 \in \mathbb{R}^d$ and $R > 0$, then*

$$\|u - u_R\|_{L^2(B_R)} \le cR^{(\alpha-\alpha_1)/2}\|\Gamma u\|_{L^{q_1}(B_R)}, \tag{4.5}$$

where $\alpha_1 = (2 - q_1)d/q_1$.

Proof. Again we may suppose $R = 1$ and obtain the general case by a scaling argument. Take $\alpha_1 < \alpha$ and let $q_1 = 2d/(d+\alpha_1)$. Note that $q_1 \in (1,2)$. By Lemma 4.1

$$\|u - u_R\|_{L^2(B_R)} \le c\Big[\int_{B_R}\int_{B_R} \frac{(u(y)-u(x))^{q_1}}{|x-y|^{d+\alpha_1 q_1/2}}\,dy\,dx\Big]^{\frac{1}{q_1}}. \tag{4.6}$$

Fix x for the moment. Using Hölder's inequality with respect to the measure $|x-y|^{-d}\,dy$,

$$\begin{aligned}
&\int_{B_R} \frac{(u(y)-u(x))^{q_1}}{|x-y|^{d+\alpha_1 q_1/2}}\,dy \\
&\quad= \int_{B_R} \frac{(u(y)-u(x))^{q_1}}{|x-y|^{\alpha q_1/2}} \frac{1}{|x-y|^{(\alpha_1-\alpha)q_1/2}} \frac{1}{|x-y|^d}\,dy \\
&\quad\le \Big[\int_{B_R} \Big(\frac{(u(y)-u(x))^{q_1}}{|x-y|^{\alpha q_1/2}}\Big)^{\frac{2}{q_1}} \frac{1}{|x-y|^d}\,dy\Big]^{\frac{q_1}{2}} \\
&\qquad\qquad \times \Big[\int_{B_R} \Big(\frac{1}{|x-y|^{(\alpha_1-\alpha)q_1/2}}\Big)^{\frac{2}{2-q_1}} \frac{1}{|x-y|^d}\,dy\Big]^{\frac{2-q_1}{2}} \\
&\quad= \Big[\int_{B_R} \frac{(u(y)-u(x))^2}{|x-y|^{d+\alpha}}\,dy\Big]^{\frac{q_1}{2}} \Big[\int_{B_R} \frac{1}{|x-y|^{(\alpha_1-\alpha)\frac{q_1}{2-q_1}+d}}\,dy\Big]^{\frac{2-q_1}{2}} \\
&\quad\le c\Big[\int_{B_R} \frac{(u(y)-u(x))^2}{|x-y|^{d+\alpha}}\,dy\Big]^{\frac{q_1}{2}} \\
&\quad\le c|\Gamma u(x)|^{q_1}.
\end{aligned}$$

Integrating over $x \in B_R$, taking the q_1^{th} root, and combining with (4.6) yields (4.5). □

Proposition 4.3. *There exists $p \in (2, 4d/(2d-\alpha))$ and a constant c_1 depending on Λ, d, α, and p such that*

$$\|\Gamma u\|_{L^p(\mathbb{R}^d)} \le c_1\Big(\mathcal{E}(u,u)^{\frac{1}{2}} + \|h\|_{L^2(\mathbb{R}^d)} + \|u\|_{L^p(\mathbb{R}^d)} + \|u\|_{L^{2p/(4-p)}(\mathbb{R}^d)}\Big).$$

Proof. Choose $x_0 \in \mathbb{R}^d$ and set $R = 1$ for now. From (4.1) we know that

$$\begin{aligned}
\|\Gamma u\|^2_{L^2(B_{R/2})} &\le c \int_{\mathbb{R}^d} (u(x) - u_R)^2 \psi(x)\, dx + \int_{B_R} |h(x)(u(x) - u_R)|\, dx \\
&\le c \int_{B_R} (u(x) - u_R)^2\, dx + c \int_{B_R^c} u(x)^2 \psi(x)\, dx \\
&\quad + c \int_{B_R^c} u_R^2 \psi(x)\, dx + \int_{B_R} |h(x)(u(x) - u_R)|\, dx \\
&= J_1 + J_2 + J_3 + J_4. \qquad (4.7)
\end{aligned}$$

We proceed to bound J_1, J_2, J_3, and J_4.

Using Proposition 4.2, we have

$$J_1 = \int_{B_R} (u(x) - u_R)^2\, dx \le c\Big(\int_{B_R} \Gamma u(x)^{q_1}\, dx\Big)^{\frac{2}{q_1}} \qquad (4.8)$$

for $q_1 \in (1, 2)$.

Note that $\psi(x) = 1 \wedge \frac{1}{|x-x_0|^{d+\alpha}}$ when $R = 1$. For any $y \in B_R$ and $x \in B_R^c$, we have $|x - y| < 2|x - x_0|$. Letting $\rho(x) = 1 \wedge \frac{1}{|x|^{d+\alpha}}$, we observe that

$$\begin{aligned}
J_2 = \int_{B_R^c} u(x)^2 \psi(x)\, dx &\le c \int_{B_R^c} u(x)^2 \Big(1 \wedge \frac{1}{|x - y|^{d+\alpha}}\Big)\, dx \\
&\le c\big((u^2) * \rho\big)(y).
\end{aligned}$$

Using Theorem 2 in Section 2.2 of Chapter 3 in [28], it follows that

$$\begin{aligned}
J_2 \le c\big((u^2) * \rho\big)(y) &\le c\Big(\int_{\mathbb{R}^d} \rho(x)\, dx\Big)\ M(u^2)(y) \\
&\le c\ M(u^2)(y),
\end{aligned}$$

where M is the Hardy-Littlewood maximal operator:

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B(x,r)} |f(y)|\, dy.$$

For any $y \in B_R$, by Jensen's inequality

$$\begin{aligned} u_R^2 &= \Big(\frac{1}{|B_R|}\int_{B_R} u(x)\,dx\Big)^2 \le \frac{1}{|B_R|}\int_{B_R} u(x)^2\,dx \\ &\le \frac{|B_{2R}|}{|B_R|}\cdot\frac{1}{|B_{2R}|}\int_{B(y,2R)} u(x)^2\,dx \\ &\le 2^d M(u^2)(y). \end{aligned}$$

Hence

$$J_3 = \int_{B_R^c} u_R^2 \psi(x)\,dx \le cM(u^2)(y)\int_{B_R^c}\psi(x)\,dx \le cM(u^2)(y).$$

Similarly, $|u_R| \le cMu(x)$ for all $x \in B_R$. Since $|B(x,s)|^{-1}\int_{B(x,s)} u(y)\,dy$ converges to $u(x)$ as $s \to 0$ for almost every x and is bounded by $Mu(x)$, we have $|u(x)| \le Mu(x)$ a.e. Thus

$$\begin{aligned} J_4 = \int_{B_R} |h(x)(u(x)-u_R)|\,dx &\le \int_{B_R}|h(x)u(x)|\,dx + \int_{B_R}|h(x)Mu(x)|\,dx \\ &\le c\int_{B_R}|h(x)|Mu(x)\,dx. \end{aligned}$$

Combining our bounds for J_1, J_2, J_3, and J_4,

$$\begin{aligned} \|\Gamma u\|^2_{L^2(B_{R/2})} \le c\|\Gamma u\|^2_{L^{q_1}(B_R)} &+ cM(u^2)(y) \\ &+ c\int_{B_R}|h(x)|Mu(x)\,dx. \end{aligned} \tag{4.9}$$

Integrating both sides of (4.9) over $y \in B_R$, we conclude that

$$\begin{aligned} \int_{B_{R/2}} \Gamma u(x)^2\,dx \le c\Big(&\int_{B_R}\Gamma u(x)^{q_1}\,dx\Big)^{\frac{2}{q_1}} \\ &+ c\int_{B_R} M(u^2)(x)\,dx + c\int_{B_R}|h(x)|Mu(x)\,dx. \end{aligned} \tag{4.10}$$

Let

$$g(x) = \Gamma u(x)^{q_1}$$

and

$$f(x) = \Big(M(u^2)(x) + |h(x)|Mu(x)\Big)^{\frac{q_1}{2}}.$$

We can rewrite (4.10) as

$$\frac{1}{|B_R|}\int_{B_{R/2}} g^{\frac{2}{q_1}}(x)\,dx \le c\Big(\frac{1}{|B_R|}\int_{B_R} g(x)\,dx\Big)^{\frac{2}{q_1}} + c\frac{1}{|B_R|}\int_{B_R} f^{\frac{2}{q_1}}(x)\,dx. \tag{4.11}$$

By a scaling and translation argument, (4.11) holds for all $R > 0$ and all $x_0 \in \mathbb{R}^d$.

We now apply the reverse Hölder inequality (see Theorem 4.1 in [12]). Thus there exists $\varepsilon > 0$ and $c_1 > 0$ such that $g(x) \in L^t(B_{R/2})$ for all $t \in [\frac{2}{q_1}, \frac{2}{q_1} + \varepsilon)$ and

$$\Big(\frac{1}{|B_{R/2}|}\int_{B_{R/2}} g^t(x)\,dx\Big)^{\frac{1}{t}} \le c\Big(\frac{1}{|B_R|}\int_{B_R} g^{\frac{2}{q_1}}(x)\,dx\Big)^{\frac{q_1}{2}} + c\Big(\frac{1}{|B_R|}\int_{B_R} f^t(x)\,dx\Big)^{\frac{1}{t}}.$$

This leads to

$$\Big(\frac{1}{|B_{R/2}|}\int_{B_{R/2}} \Gamma u(x)^{q_1 t}\,dx\Big)^{\frac{1}{t}} \le c\Big(\frac{1}{|B_R|}\int_{B_R} \Gamma u(x)^2\,dx\Big)^{\frac{q_1}{2}} + c\Big(\frac{1}{|B_R|}\int_{B_R} (M(u^2))^{tq_1/2}(x)\,dx\Big)^{\frac{1}{t}} + c\Big(\frac{1}{|B_R|}\int (|h|Mu)^{tq_1/2}\,dx\Big)^{1/t}.$$

Choose $t \in (2/q_1, 2/q_1 + \varepsilon)$ so that $q_1 t < 4d/(d-\alpha)$ and set $p = q_1 t$.

Now set $R = 2\sqrt{d}$ for the remainder of the proof. Taking q_1^{th} roots and using the inequality $(a+b)^{1/q_1} \le a^{1/q} + b^{1/q_1}$,

$$\|\Gamma u\|_{L^p(B_{R/2})} \le c\|\Gamma u\|_{L^2(B_R)} + c\|M(u^2)\|^{1/2}_{L^{p/2}(B_R)} + c\|h(Mu)\|^{1/2}_{L^{p/2}(B_R)}.$$

For $k \in \mathbb{Z}^d$, let $C_k = B(k, \sqrt{d})$ and $D_k = B(k, 2\sqrt{d})$. Note that $\mathbb{R}^d \subset \cup_{k \in \mathbb{Z}^d} C_k$ and that there exists an integer N depending only on the dimension d such that no point of $\mathbb{R}^d$ is in more than N of the D_k. This can be expressed as $\sum_{k \in \mathbb{Z}^d} \chi_{D_k} \le N$.

Using $\sum a_k^{p/2} \le (\sum a_k)^{p/2}$ when each $a_k \ge 0$ and $p/2 \ge 1$, we write

$$
\begin{aligned}
\int_{\mathbb{R}^d} |\Gamma u(x)|^p \, dx &\le \sum_{k \in \mathbb{Z}^d} \int_{C_k} |\Gamma u(x)|^p \, dx \\
&\le c \sum_k \Big(\int_{D_k} |\Gamma u(x)|^2 \, dx \Big)^{p/2} + c \sum_k \int_{D_k} (M(u^2)(x))^{p/2} \, dx \\
&\qquad + c \sum_k \int_{D_k} (|h(x)| M u(x))^{p/2} \, dx \\
&\le c \Big(\sum_k \int_{D_k} |\Gamma u(x)|^2 \, dx \Big)^{p/2} + c \sum_k \int_{D_k} (M(u^2)(x))^{p/2} \, dx \\
&\qquad + c \sum_k \int_{D_k} (|h(x)| M u(x))^{p/2} \, dx \\
&= c \Big(\int_{\mathbb{R}^d} |\Gamma u(x)|^2 \sum_k \chi_{D_k}(x) \, dx \Big)^{p/2} \\
&\qquad + c \int_{\mathbb{R}^d} (M(u^2)(x))^{p/2} \sum_k \chi_{D_k}(x) \, dx \\
&\qquad + c \int_{\mathbb{R}^d} (|h(x)| M u(x))^{p/2} \sum_k \chi_{D_k}(x) \, dx.
\end{aligned}
$$

We thus obtain

$$
\begin{aligned}
\int_{\mathbb{R}^d} |\Gamma u|^p \le c \Big(\int_{\mathbb{R}^d} |\Gamma u|^2 \, dx \Big)^{p/2} &+ c \int_{\mathbb{R}^d} (M(u^2))^{p/2} \, dx \\
&+ c \int_{\mathbb{R}^d} (|h| \, M u)^{p/2} \, dx. \qquad (4.12)
\end{aligned}
$$

Letting $r = 4/p$ and $s = 4/(4-p)$, Hölder's inequality and the inequality

$ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$ shows

$$\begin{aligned}\int (|h| Mu)^{p/2} &\le \Big(\int |h|^{pr/2}\Big)^{1/r}\Big(\int (Mu)^{ps/2}\Big)^{1/s}\\ &\le \tfrac{1}{2}\Big(\int |h|^2\Big)^{p/2} + \tfrac{1}{2}\Big(\int (Mu)^{2p/(4-p)}\Big)^{(4-p)/2}. \qquad (4.13)\end{aligned}$$

Since M is a bounded operator on $L^{p'}$ for each $p' > 1$ and we know that $2p/(4-p) > 1$, the second term on the last line of (4.13) is bounded by

$$c\Big(\int |u|^{2p/(2-p)}\Big)^{(4-p)/2}.$$

Similarly, since $p > 2$, the second term on the right hand side of the first line of (4.12) is bounded by

$$c\int (|u|^2)^{p/2} = c\int |u|^p.$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}^d} |\Gamma u|^p \le c\Big(\int |\Gamma u|^2\Big)^{p/2} + c\int |u|^p + c\Big(\int |h|^2\Big)^{p/2}\\ + c\Big(\int |u|^{2p/(4-p)}\Big)^{(4-p)/2}.\end{aligned}$$

Taking p^{th} roots and using $(a+b)^{1/p} \le a^{1/p} + b^{1/p}$, we obtain

$$\begin{aligned}\|\Gamma u\|_{L^p(\mathbb{R}^d)} \le c\|\Gamma u\|_{L^2(\mathbb{R}^d)} + c\|u\|_{L^p(\mathbb{R}^d)} + c\|h\|_{L^2(\mathbb{R}^d)}\\ + c\|u\|_{L^{2p/(4-p)}(\mathbb{R}^d)}.\end{aligned}$$

This completes the proof of the proposition. □

We now bound the L^p and $L^{2p/(4-p)}$ norms of u.

Theorem 4.4. *(1) Suppose $d > \alpha$ and (3.2) holds. There exists $p > 2$ and a constant c_1 depending on Λ, p, d, and α such that*

$$\|\Gamma u\|_{L^p(\mathbb{R}^d)} \le c_1\Big(\mathcal{E}(u,u)^{\frac{1}{2}} + \|h\|_{L^2(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)}\Big).$$

(2) If in addition $u \in \mathcal{D}(\mathcal{L})$, there exists a constant c_2 such that

$$\|\Gamma u\|_{L^p(\mathbb{R}^d)} \le c_1\Big(\|h\|_{L^2(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)}\Big).$$

Proof. Let $p_1 = 2d/(d-\alpha)$. Let C_k be defined as in the previous proof.

By Lemma 4.1 with $q = 2$

$$\int_{C_k} |u - u_{C_k}|^{p_1} \le c\Big(\int_{C_k} |\Gamma u(x)|^2\,dx\Big)^{p_1/2}.$$

Here $u_{C_k} = (1/|C_k|)\int_{C_k} u$. Then

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d}\int_{C_k} |u - u_{C_k}|^{p_1} &\le c\sum_k \Big(\int_{C_k} |\Gamma u(x)|^2\,dx\Big)^{p_1/2}\\ &\le c\Big(\sum_k \int_{C_k} |\Gamma u(x)|^2\,dx\Big)^{p_1/2}\\ &\le c\Big(\int_{\mathbb{R}^d} |\Gamma u(x)|^2 \sum_k \chi_{C_k}(x)\,dx\Big)^{p_1/2}\\ &\le c\Big(\int_{\mathbb{R}^d} |\Gamma u(x)|^2\,dx\Big)^{p_1/2}.\end{aligned}$$

Also,

$$\int_{C_k} |u_{C_k}|^{p_1} = c|u_{C_k}|^{p_1} \le c\Big(\int_{C_k} |u|^2\Big)^{p_1/2}$$

by Jensen's inequality. Similarly to the above,

$$\sum_k \int_{C_k} |u_{C_k}|^{p_1} \le c\Big(\int_{\mathbb{R}^d} u^2\Big)^{p_1/2}.$$

Hence

$$\begin{aligned}\int |u|^{p_1} \le \sum_k \int_{C_k} |u|^{p_1} &\le c\sum_k \int_{C_k} |u - u_{C_k}|^{p_1} + \sum_k \int_{C_k} |u_{C_k}|^{p_1}\\ &\le c\Big(\int |\Gamma u|^2\Big)^{p_1/2} + c\Big(\int u^2\Big)^{p_1/2}.\end{aligned}$$

Taking ${p_1}^{th}$ roots, we have

$$\|u\|_{L^{p_1}(\mathbb{R}^d)} \le c\|\Gamma u\|_{L^2(\mathbb{R}^d)} + c\|u\|_{L^2(\mathbb{R}^d)}.$$

If $2 \le r \le p_1$, there exists $\theta \in [0,1]$ depending only on r and p_1 such that $\|u\|_{L^r} \le \|u\|_{L^2}^{\theta}\|u\|_{L^{p_1}}^{1-\theta}$; see, e.g., Proposition 6.10 of [20]. Combining with the inequality $a^\theta b^{1-\theta} \le a + b$ yields

$$\|u\|_{L^r} \le \|u\|_{L^2} + \|u\|_{L^{p_1}}.$$

We thus obtain

$$\|u\|_{L^r(\mathbb{R}^d)} \leq c\|\Gamma u\|_{L^2(\mathbb{R}^d)} + c\|u\|_{L^2(\mathbb{R}^d)}.$$

Applying this with r first equal to p and then with r equal to $2p/(4-p)$, we obtain (1).

Suppose now that $u \in \mathcal{D}(\mathcal{L})$ and that $h = \mathcal{L}u$. Let $\{E_\lambda\}$ be the spectral resolution of the operator $-\mathcal{L}$. Then for $u \in L^2$,

$$u = \int_0^\infty dE_\lambda u, \qquad \|u\|_{L^2(\mathbb{R}^d)} = \int_0^\infty d(E_\lambda u, E_\lambda u).$$

If $u \in \mathcal{D}(\mathcal{L})$ and $h = \mathcal{L}u$, then

$$h = \int_0^\infty \lambda\, dE_\lambda u, \qquad \|h\|_{L^2(\mathbb{R}^d)} = \int_0^\infty \lambda^2\, d(E_\lambda u, E_\lambda u).$$

It then follows that

$$\begin{aligned}
\|\Gamma u\|^2_{L^2(\mathbb{R}^d)} &= \mathcal{E}(u,u) \\
&= \int_0^\infty \lambda\, d(E_\lambda u, E_\lambda u) \\
&= \int_0^1 \lambda\, d(E_\lambda u, E_\lambda u) + \int_1^\infty \lambda\, d(E_\lambda u, E_\lambda u) \\
&\leq \int_0^1 d(E_\lambda u, E_\lambda u) + \int_1^\infty \lambda^2\, d(E_\lambda u, E_\lambda u) \\
&\leq \|u\|^2_{L^2(\mathbb{R}^d)} + \|h\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

This proves (2). □

5 Strong stability

Let

$$G(x) = \sup_{y \in \mathbb{R}^d} |\widetilde{A}(x,y) - A(x,y)|.$$

Theorem 5.1. *Suppose $d > \alpha$. For any $f(x) \in L^2(\mathbb{R}^d)$ and $q \geq 2d/\alpha$, there exists a constant c_1 depending on Λ, d, α, and q such that*

$$\|P_t f - \widetilde{P}_t f\|_{L^2}^2 \leq c_1\ t^{-\frac{1}{2}} \|G\|_{L^{2q}} \|f\|_{L^2}^2.$$

Proof. For $t > 0$, let $u = P_t f - \widetilde{P}_t f$. By Lemma 2.1, we know that $P_t f$ and $\widetilde{P}_t f$ are both in $\mathcal{F} = W^{\frac{\alpha}{2},2}(\mathbb{R}^d)$, so $u \in W^{\frac{\alpha}{2},2}(\mathbb{R}^d)$.

We write

$$\begin{aligned}\|P_t f - \widetilde{P}_t f\|_{L^2}^2 &= (P_t f - \widetilde{P}_t f, u) \\ &= \int_0^t \frac{d}{ds}(P_s \widetilde{P}_{t-s} f, u)\, ds.\end{aligned}$$

This, Lemma 2.3, and routine calculations show that

$$\|P_t f - \widetilde{P}_t f\|_{L^2}^2 = \int_0^t \Big(- \mathcal{E}(\widetilde{P}_{t-s} f, P_s u) + \widetilde{\mathcal{E}}(\widetilde{P}_{t-s} f, P_s u)\Big)\, ds. \tag{5.1}$$

Using (5.1), Lemma 2.1 and Hölder's inequality, we obtain

$$\begin{aligned}&\|P_t f - \widetilde{P}_t f\|_{L^2}^2 \\ &\quad = \int_0^t \Big(- \mathcal{E}(\widetilde{P}_{t-s} f, P_s u) + \widetilde{\mathcal{E}}(\widetilde{P}_{t-s} f, P_s u)\Big)\, ds \\ &\quad = \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Big(\widetilde{P}_{t-s} f(y) - \widetilde{P}_{t-s} f(x)\Big)\Big(P_s u(y) - P_s u(x)\Big) \\ &\qquad\quad \times \frac{\widetilde{A}(x,y) - A(x,y)}{|x-y|^{d+\alpha}}\, dy\, dx\, ds \\ &\quad \leq c \int_0^t \Big[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Big(\widetilde{P}_{t-s} f(y) - \widetilde{P}_{t-s} f(x)\Big)^2 \frac{1}{|x-y|^{d+\alpha}}\, dy\, dx\Big]^{\frac{1}{2}} \\ &\qquad\quad \times \Big[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Big(P_s u(y) - P_s u(x)\Big)^2 \frac{|\widetilde{A}(x,y) - A(x,y)|^2}{|x-y|^{d+\alpha}}\, dy\, dx\Big]^{\frac{1}{2}} ds \\ &\quad \leq c \int_0^t \Big[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Big(\widetilde{P}_{t-s} f(y) - \widetilde{P}_{t-s} f(x)\Big)^2 \frac{\widetilde{A}(x,y)}{|x-y|^{d+\alpha}}\, dy\, dx\Big]^{\frac{1}{2}} \\ &\qquad\quad \times \Big[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Big(P_s u(y) - P_s u(x)\Big)^2 \frac{|\widetilde{A}(x,y) - A(x,y)|^2}{|x-y|^{d+\alpha}}\, dy\, dx\Big]^{\frac{1}{2}} ds\end{aligned}$$

$$
\begin{aligned}
&\leq c\int_0^t \left[\widetilde{\mathcal{E}}(\widetilde{P}_{t-s}f,\widetilde{P}_{t-s}f)\right]^{\frac{1}{2}}\\
&\qquad \times \Big[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\Big(P_su(y)-P_su(x)\Big)^2}{|x-y|^{d+\alpha}}\,dy\ G^2(x)\,dx\Big]^{\frac{1}{2}}\,ds\\
&\leq c\int_0^t (t-s)^{-\frac{1}{2}}\|f\|_{L^2}\\
&\qquad \times \Big\{\int_{\mathbb{R}^d}\Big[\int_{\mathbb{R}^d}\frac{\Big(P_su(y)-P_su(x)\Big)^2}{|x-y|^{d+\alpha}}\,dy\Big]^{p'}\,dx\Big\}^{\frac{1}{2p'}} \qquad (5.2)\\
&\qquad \times \Big\{\int_{\mathbb{R}^d} G^{2q'}(x)\,dx\Big\}^{\frac{1}{2q'}}\,ds\\
&= c\|f\|_{L^2}\|G\|_{L^{2q'}}\int_0^t (t-s)^{-\frac{1}{2}}\|\Gamma(P_su)(x)\|_{L^{2p'}}\,ds, \qquad (5.3)
\end{aligned}
$$

where p' and q' are conjugate exponents.

We choose p' so that $2p'$ is equal to the p in Theorem 4.4. By that theorem,

$$
\|\Gamma(P_su)\|_{L^{2p'}} \leq c\|P_su\|_{L^2} + c\|\mathcal{L}(P_su)\|_{L^2}. \qquad (5.4)
$$

Since P_s, P_t, and $\widetilde{P}_t$ are contractions,

$$
\|P_su\|_{L^2} \leq \|u\|_{L^2} = \|P_tf - \widetilde{P}_tf\|_{L^2} \leq 2\|f\|_{L^2}. \qquad (5.5)
$$

To estimate $\mathcal{L}(P_su)$, we note $P_{s/2}u \in \mathcal{D}(\mathcal{L})$ by Lemma 2.2 and we use Lemma 2.4. Then

$$
\begin{aligned}
\|\mathcal{L}(P_su\|_{L^2} &= \|(-\mathcal{L})^{1/2}P_{s/2}(-\mathcal{L})^{1/2}(P_{s/2}u)\|_{L^2} \qquad (5.6)\\
&\leq cs^{-1/2}\|(-\mathcal{L})^{1/2}(P_{s/2}u)\|_{L^2}\\
&= cs^{-1/2}\mathcal{E}(P_{s/2}u,P_{s/2}u)^{1/2}\\
&\leq cs^{-1/2}\mathcal{E}(u,u)^{1/2}\\
&\leq cs^{-1/2}\big[\mathcal{E}(P_tf,P_tf)^{1/2}+\mathcal{E}(\widetilde{P}_tf,\widetilde{P}_tf)^{1/2}\big]\\
&\leq c(st)^{-1/2}\|f\|_{L^2}.
\end{aligned}
$$

Combining (5.3), (5.4), (5.5), and (5.6) yields our result. □

Let $p(t,x,y)$ and $\widetilde{p}(t,x,y)$ be the heat kernels corresponding to P_t and $\widetilde{P}_t$. By Theorem 4.14 in [15], we know there exist $\gamma > 0$ and a constant c_1 such that

$$|p(t,x,y) - p(t,z,v)| \le c_1 \, t^{-\frac{d+\gamma}{\alpha}} (|x-z| + |y-v|)^\gamma \tag{5.7}$$

for all $x, y, z, v \in \mathbb{R}^d$. By Theorem 1.1 in [15], there exist constants c_2 and c_3 such that

$$\begin{aligned} c_2 \min\left\{t^{-\frac{d}{\alpha}}, \tfrac{t}{|x-y|^{d+\alpha}}\right\} &\le p(t,x,y) \\ &\le c_3 \min\left\{t^{-\frac{d}{\alpha}}, \frac{t}{|x-y|^{d+\alpha}}\right\} \end{aligned} \tag{5.8}$$

for all $x, y \in \mathbb{R}^d$.

We have the following two theorems. Once we have Theorem 5.1, (5.7), and (5.8), the proofs are so similar to the corresponding theorems in [18] that we refer the reader to that paper for the proofs.

Theorem 5.2. *For $q > 1$ and $t > 0$, there exists a constant c_1 depending on Λ, d, α, and q such that for any $x, y \in \mathbb{R}^d$*

$$|p(t,x,y) - \widetilde{p}(t,x,y)| \le c_1 \min\left\{t^{-\frac{d}{\alpha}}, \frac{t}{|x-y|^{d+\alpha}}, t^{-\frac{d}{\alpha} - \frac{\gamma}{4(d+\gamma)}} \; \|G(x)\|_{2q}^{\frac{\gamma}{2(d+\gamma)}}\right\}.$$

Theorem 5.3. *For $q > 1$ and $t > 0$, there exists a constant $c_1 = c_1(\lambda, d, \alpha, q)$ such that for any $p \in [1, \infty]$, we have*

$$\|P_t f - \widetilde{P}_t f\|_{L^p} \le \min\left\{2, c_1 \left\{t^{-\frac{1}{4}} \; \|G(x)\|_{2q}^{\frac{1}{2}}\right\}^{\frac{\gamma\alpha}{(d+\gamma)(d+\alpha)}}\right\} \|f\|_{L^p}.$$

References

[1] R.A. Adams, *Sobolev spaces*, New York, Academic Press, 1975.

[2] M.T. Barlow, A. Grigor'yan and T. Kumagai, Heat kernel upper bounds for jump processes and the first exit time. *J. Reine Angew. Math.* **626** (2006) 135-157.

[3] M.T. Barlow, R.F. Bass, Z.-Q. Chen, and M. Kassmann, Non-local Dirichlet forms and symmetric jump processes, *Trans. Amer. Math. Soc.* **361** (2009) 1963–1999.

[4] M.T. Barlow, A. Grigor'yan and T. Kumagai, On the equivalence of parabolic Harnack inequalities and heat kernel estimates. *J. Math. Soc. Japan*, to appear.

[5] R.F. Bass, Uniqueness in law for pure jump Markov processes. *Probab. Theory rel. Fields* **79** (1988) 271–287.

[6] R.F. Bass and D.A. Levin, Harnack inequalities for jump processes. *Potential Anal.* **17** (2002) 375–388.

[7] R.F. Bass and D.A. Levin, Transition Probabilities for symmetric jump processes. *Trans.Amer.Math. Soc.* **354** (2002) 2933-2953.

[8] R.F. Bass and M. Kassmann, Harnack inequalities for non-local operators of variable order. *Trans. Amer. Math. Soc.* **357** (2005) 837–850.

[9] R.F. Bass and M. Kassmann, Hölder continuity of harmonic functions with respect to operators of variable order, *Comm. PDE* **30** (2005) 1249–1259.

[10] R.F. Bass and H. Tang, The martingale problem for a class of stable-like processes, *Stoch. Proc. and their Applic.* **119** (2009) 1144–1167.

[11] R.F. Bass, M. Kassmann, and T. Kumagai, Symmetric jump processes: localization, heat kernels, and convergence, *Ann. de l'Institut H. Poincaré* **46** (2010) 59–71.

[12] Y.-Z. Chen and L.-C. Wu, Second order elliptic systems and equations, AMS , Providence, 1992.

[13] Z.-Q.Chen, P. Kim, and T. Kumagai, Global heat kernel estimates for symmetric jump processes, *Trans. Amer. Math. Soc.* **363** (2011) 5021–5055.

[14] Z.-Q. Chen, P. Kim, and R. Song, Two-sided heat kernel estimates for censored stable-like processes. *Probab. Th. rel. Fields* **146** (2010) 361-399.

[15] Z.-Q. Chen and T. Kumagai, Heat kernel estimate for stable -like processes on d-sets. *Stoch. processes and their Applic.* **108** (2003) 27–62.

[16] Z.-Q. Chen and T. Kumagai, Heat kernel estimates for jump processes of mixed types on metric measure spaces. *Probab. Theory Relat. Fields* **140** (2008) 277–317.

[17] Z.-Q. Chen and T. Kumagai, A priori Hölder estimate, parabolic Harnack principle and heat kernel estimates for diffusions with jumps, *Rev. Mat. Iberoamericana,* **26** (2010) 551–589.

[18] Z-Q. Chen, Z. Qian, Y. Hu and W. Zheng, Stability and approximations of symmetric diffusion semigroups and kernels. *J. Functional Analysis* **152** (1998) 255–280.

[19] D.E. Edmunds and H. Triebel, *Functions Spaces, Entropy Numbers, Differential Operators*, Cambridge University Press, 1996.

[20] G.B. Folland, *Real Analysis: Modern Techniques and their Applications,* 2^{nd} *ed.*, Wiley, New York, 1999.

[21] M. Foondun, Harnack inequalities for a class of integro-differential operators. *Potential Anal.* **31** (2009) 21–44.

[22] M. Foondun, Heat kernel estimates and Harnack inequalities for some Dirichlet forms with non-local parts. *Electron. J. Probab.* **14** (2009), Paper No. 11, 314–340.

[23] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, de Gruyter, Berlin, 1994.

[24] V. Kolokoltsov, Symmetric stable laws and stable-like jump-diffusions. *Proc. London Math. Soc.* **80** (2000) 725–768.

[25] N.G. Meyers, An L^p-estimate for the gradient of solutions of second order elliptic divergence equations , *Ann. Scuola Norm. Sup. Pisa* **17** (1963) 189–206.

[26] G. Mingione, The singular set of solutions to non-differentiable elliptic systems. *Arch. Rational Mech. Anal.* **166** (2003), 287–301.

[27] R. Song and Z. Vondraček, Harnack inequality for some classes of Markov processes. *Math. Z.* **246** (2004) 177–202.

[28] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, 1970.

Richard F. Bass
Department of Mathematics
University of Connecticut
Storrs, CT 06269-3009, USA
r.bass@uconn.edu

Hua Ren
Department of Mathematics
University of Connecticut
Storrs, CT 06269-3009, USA
hua.ren@uconn.edu