SYSTEMS OF EQUATIONS DRIVEN BY STABLE PROCESSES

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Abstract. Let Z_t^j, $j=1,\ldots,d$, be independent one-dimensional symmetric stable processes of index $\alpha\in(0,2)$. We consider the system of stochastic differential equations

$$dX_t^i=\sum_{j=1}^d A_{ij}(X_{t-})dZ_t^j, \qquad i=1,\ldots,d,$$

where the matrix $A(x)=(A_{ij}(x))_{1\le i,j\le d}$ is continuous and bounded in x and nondegenerate for each x. We prove existence and uniqueness of a weak solution to this system. The approach of this paper uses the martingale problem method. For this, we establish some estimates for pseudodifferential operators with singular state-dependent symbols. Let $\lambda_2>\lambda_1>0$. We show that for any two vectors $a,b\in\mathbb{R}^d$ with $|a|,|b|\in(\lambda_1,\lambda_2)$ and p sufficiently large, the L^p-norm of the operator whose Fourier multiplier is $(|u\cdot a|^\alpha-|u\cdot b|^\alpha)/\sum_{j=1}^d|u_i|^\alpha$ is bounded by a constant multiple of $|a-b|^\theta$ for some $\theta>0$, where $u=(u_1,\ldots,u_d)\in\mathbb{R}^d$. We deduce from this the L^p-boundedness of pseudodifferential operators with symbols of the form $\psi(x,u)=|u\cdot a(x)|^\alpha/\sum_{j=1}^d|u_i|^\alpha$, where $u=(u_1,\ldots,u_d)$ and a is a continuous function on $\mathbb{R}^d$ with $|a(x)|\in(\lambda_1,\lambda_2)$ for all $x\in\mathbb{R}^d$.

Keywords. Stable processes, stochastic differential equations, martingale problem, weak solution, weak uniqueness, pseudodifferential operators, method of rotations

Short title. Systems of SDEs

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1. Introduction.

Let $Z_t^1, \ldots, Z_t^d$ be independent one-dimensional symmetric stable processes of index α for some $\alpha \in (0,2)$. Consider the system of stochastic differential equations

$$dX_t^i = \sum_{j=1}^d A_{ij}(X_{t-})dZ_t^j, \qquad i = 1, \ldots, d, \quad \text{with } X_0 = x_0. \tag{1.1}$$

Here $d \geq 2$ and $x_0 \in \mathbb{R}^d$. We assume that the $A_{ij}(x)$, $1 \leq i,j \leq d$, are continuous and bounded as functions of x and that for each $x \in R^d$ the matrix $A(x) = (A_{ij}(x))$ is nondegenerate. The purpose of this paper is to prove the following.

Theorem 1.1. *For each $x_0 \in \mathbb{R}^d$, there exists one and only one weak solution*

$$\left\{ X = \{(X_t^1, \ldots, X_t^d),\, t \geq 0\},\ \mathbb{P}^{x_0} \right\}$$

to the SDE (1.1). The family $\{X,\ \mathbb{P}^x,\ x \in \mathbb{R}^d\}$ forms a conservative strong Markov process on $\mathbb{R}^d$ whose semigroup maps bounded continuous functions to bounded continuous functions.

Roughly speaking, the uniqueness of a weak solution means that the law of X is uniquely determined. See Section 2 for a precise definition.

In the last few years there has been intensive interest in the study of processes with jumps. Much of the motivation has come from mathematical physics and from financial mathematics: in many applications jump processes provide more realistic models than continuous processes do. The SDE system (1.1) is the exact analogue for stable processes to the system of equations used in defining continuous diffusions and is a very natural object of study.

A particular case of the system (1.1) (with dimension $d = 2$) arose in [BBC] in the study of the pathwise uniqueness of a 1-dimensional SDE driven by a symmetric stable process. There we were able to sidestep the issue of uniqueness, but that problem also motivated us in the present work.

In this paper, we concentrate on the existence of weak solutions to SDE (1.1) and investigate the uniqueness in law of the solution. One might ask about pathwise uniqueness. It is clear that if the coefficients $A_{ij}(x)$ are bounded Lipschitz functions, the Picard iteration method gives a strong solution to the SDE (1.1) and the solution is pathwise unique. However if the matrix-valued function $A(x) = (A_{ij}(x))$ is merely uniformly nondegenerate, bounded and continuous, pathwise uniqueness needs not to hold in general even in the case of dimension $d = 1$. The latter fact is the main result of [BBC], which combined with that of [Km2] and [B4] gives sharp results for the pathwise uniqueness of the SDE (1.1) when $d = 1$. It is an interesting question to find optimal conditions on $A_{ij}(x)$ so that pathwise uniqueness holds for the SDE (1.1) in dimensions $d \geq 2$.

In the special case $d = 1$ a time change argument can be given to give a proof of Theorem 1.1; see [BBC] for details. So in this paper we will concentrate on the case of dimension $d \geq 2$.

It is instructive to consider the difference between the process $Z_t := (Z_t^1, Z_t^2, \cdots, Z_t^d)$ that consists of d independent copies of one-dimensional symmetric α-stable processes and a d-dimensional spherically symmetric α-stable process V_t. Both processes are multidimensional Lévy processes. The process V_t picks a direction at random (uniformly over the unit sphere) and then jumps a distance in that direction. So the Lévy measure is absolutely continuous with respect to d-dimensional Lebesgue measure on $\mathbb{R}^d$. By contrast, Z_t picks a coordinate at random (from $\{1, \dots, d\}$) and then jumps a positive or negative distance in that direction. The Lévy measure for Z is concentrated on the union of the coordinate axes, a 1-dimensional subset of $\mathbb{R}^d$. Thus Z (and hence the system (1.1)) is a much more singular case than if we were considering stochastic differential equations driven by the coordinates of V. Martingale problems related to V have provided the motivation for some of the literature in this subject (e.g., [B2], [Km3], [St], [TTW]), and are much easier to deal with (see Section 7 below for details).

We use truncation and weak convergence methods to show that for every $x_0 \in \mathbb{R}^d$ there is a weak solution to (1.1). For the weak uniqueness of solutions to (1.1), we consider the related martingale problem (see Section 2 for a precise definition). Define the operator $\mathcal{L}$ by

$$\mathcal{L}f(x) = \sum_{j=1}^d \int_{\mathbb{R}\setminus\{0\}} \big(f(x + a_j(x)w) - f(x) - w 1_{\{|w|\le 1\}} \nabla f(x) \cdot a_j(x) \big) \frac{c_\alpha}{|w|^{1+\alpha}} dw. \tag{1.2}$$

Here $a_j(x)$ is the j^{th} column of the matrix $A(x)$. We will show that $\mathcal{L}$ acts as the generator for any weak solution X of (1.1) and that the weak uniqueness for the SDE (1.1) is equivalent to uniqueness for the solution to the martingale problem for $\mathcal{L}$. Fix $x_0 \in \mathbb{R}^d$. The idea of our approach is to view $\mathcal{L}$ as a perturbation of

$$\mathcal{L}_0 f(x) := \sum_{j=1}^d \int_{\mathbb{R}\setminus\{0\}} \big(f(x + a_j(x_0)w) - f(x) - w 1_{\{|w|\le 1\}} \nabla f(x) \cdot a_j(x_0) \big) \frac{c_\alpha}{|w|^{1+\alpha}} dw.$$

Note that $\mathcal{L}_0$ is the generator for the process $U_t := U_0 + A(x_0) Z_t$.

Suppose $\mathbb{P}^{x_0}$ is a weak solution to (1.1). We will use $\mathbb{E}^{\mathbb{P}^{x_0}}$ (and in the sequel, sometime just $\mathbb{E}$) to denote the expectation taken under the law of $\mathbb{P}^{x_0}$. Note that we want to solve the SDE (1.1) with a prescribed initial condition $X_0 = x_0$ and we do not know *a priori* its existence and uniqueness. So we do not know in advance if $\{X, \mathbb{P}^x, x \in \mathbb{R}^d\}$ forms a strong Markov process. Hence we avoid using $\mathbb{E}^{x_0}$ for the expectation under $\mathbb{P}^{x_0}$. On the other hand, since $U := \{U_t, t \ge 0\}$ is an affine transform of Z, which is a Lévy process, we will use $\mathbb{E}^x$ to denote the expectation with respect to the law of U when $U_0 = x$. This notation will be in force throughout this paper.

For $\lambda > 0$, define

$$S_\lambda f := \mathbb{E}^{\mathbb{P}^{x_0}} \left[\int_0^\infty e^{-\lambda t} f(X_t)\, dt \right] \quad \text{and} \quad R_\lambda f(x) := \mathbb{E}^x \left[\int_0^\infty e^{-\lambda t} f(U_t)\, dt \right].$$

Here again, since we are solving (1.1) for a particular initial condition $X_0 = x_0$ and we do not know *a priori* that S_λ defines a resolvent as x_0 ranges over $\mathbb{R}^d$. Later, after Theorem 1.1 is established, it will follow that S_λ is indeed a resolvent. Since U is a Lévy process, it is clear that R_λ is a resolvent operator and so we use $R_\lambda f(x)$ to denote the value of the function $R_\lambda f$ at $x \in \mathbb{R}^d$. Since $d \ge 2$, the process U_t is transient (cf. Theorem I.17 of [Be]) and so R_0 is well defined for $\lambda = 0$. It is not hard to see that for a smooth function f with compact support in $\mathbb{R}^d$,

$$S_\lambda f = R_\lambda f(x_0) + S_\lambda \mathcal{B} R_\lambda f, \tag{1.3}$$

where $\mathcal{B} = \mathcal{L} - \mathcal{L}_0$. To show that weak uniqueness holds for (1.1), it suffices to show that there is a unique operator S_λ satisfying (1.3). The key to doing this is to show that the L^p-norm of the operator

$$f \mapsto \sup_{|a-a_0|<\delta} |\mathcal{K}_a f - \mathcal{K}_{a_0} f|, \tag{1.4}$$

is bounded by $c_1 \delta^\theta$ for some positive constants c_1 and θ. Here $a_0 \in \mathbb{R}^d \setminus \{0\}$ and for $a \in \mathbb{R}^d$

$$\mathcal{K}_a f(x) := \int_{\mathbb{R}\setminus\{0\}} \big(R_0 f(x+ah) - R_0 f(x) - h 1_{\{|h|\le 1\}} \nabla R_0 f(x) \cdot a \big) \frac{1}{|h|^{1+\alpha}}\, dh.$$

Associated with the operator $\mathcal{L}$ is the symbol

$$\ell(x,u) := -c_2 \sum_{j=1}^d |u \cdot a_j(x)|^\alpha, \qquad x, u \in \mathbb{R}^d. \tag{1.5}$$

This means

$$\mathcal{L} f(x) = -\int_{\mathbb{R}^d} \ell(x,u) e^{-iu\cdot x} \widehat{f}(u)\, du,$$

where $\widehat{f}$ denotes the Fourier transform of f. This is an example of a pseudodifferential operator with state-dependent symbol.

The use of pseudodifferential operators in probability is not new, but the literature deals mostly with operators with smooth symbols. See [H1], [H2], [HJ], [J1], [J2], [JS], and [Kl] and the references therein for a sample of papers using them. However, the symbols in (1.5) of the pseudodifferential operators for the system of SDEs (1.1) are state-dependent and are singular along certain directions. They are not C^2 on $\mathbb{R}^d \setminus \{0\}$ in the variable u (they are not even C^1 in u when $\alpha \le 1$) and are merely continuous in the variable x. So the standard results in the theory of pseudodifferential operators such as those in Calderón and Zygmund [CZ1], [CZ2], Stein [S], and Stein and Weiss [SW] do not apply directly. In Section 3, we establish an L^p-bound on the operator in (1.4) using the method of rotations.

There have been a number of papers studying uniqueness for martingale problems corresponding to non-local operators. See [B2], [H1], [H2], [HJ], [Kl], [Km1], [Km3], [LM], [Sk], [St], [T], [TTW], and the references therein. Most of the existing literature is concerned with the cases where either the non-local operator has a nondegenerate elliptic operator component (e.g. [Km1], [LM], [St]), or the leading term in the purely non-local part is driven by the Lévy measure of a spherically symmetric stable process or its variant, which is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^d$ (e.g. [B2], [Km3], [St], [T], [TTW]). In [H1], uniqueness of the martingale problem for pseudodifferential operators with symbols of the form

$$\ell(x,u) = -\sum_{i=1}^d b_i(x) \xi_i(u) \tag{1.6}$$

are studied, where b_i, $1 \le i \le d$, are non-negative bounded C^{d+m} functions for some integer $m \ge 1$ and ξ_i, $1 \le i \le d$, are continuous non-negative definite functions on $\mathbb{R}^d$ with $\xi_i(0) = 0$

(for example, $\xi_i = |u_i|^\alpha$). The above uniqueness result was extended in [H2] to symbols $l(x,u)$ not necessarily of the form (1.6) but certain smoothness in the variable x is required. In [Kl], the uniqueness of the martingale problem for pseudodifferential operators with symbols of the form (1.6) has also been studied, where less smoothness on the $b_j(x)$ is required and the $b_j(x)$ are allowed to be unbounded. Neither of these results cover the symbols of the form (1.5). It is a bit surprising that the system of SDEs driven by d independent one-dimensional symmetric α-stable processes has not been systematically studied previously.

If the process $Z_t := (Z_t^1, Z_t^2, \cdots, Z_t^d)$ is replaced by the spherically symmetric α-stable process V_t in $\mathbb{R}^d$, then it can be shown in a way similar to Proposition 4.1 below that the generator for any weak solution of SDE (1.1) is

$$\widetilde{\mathcal{L}} f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \big(f(x + A(x)y) - f(x) - \nabla f(x) \cdot (A(x)y) 1_{\{|y| \le 1\}} \big) \frac{c_{d,\alpha}}{|y|^{d+\alpha}} \, dy,$$

whose symbol is

$$\widetilde{\ell}(x,u) = -c_3 |A(x)u|^\alpha, \qquad x, u \in \mathbb{R}^d. \tag{1.7}$$

This type of symbol is not covered in [H1]-[H2] and [Kl] either as $A(x)$ is merely continuous. The SDE (1.1) driven by a spherically symmetric α-stable process instead of by Z is easier to study since $\widetilde{\ell}(x,u)$ is homogeneous of degree α in u and is C^∞ in u in $\mathbb{R}^d \setminus \{0\}$. However the authors could not find any results in the literature about the existence or uniqueness of weak solutions for such SDE systems. In the last section, we will indicate how to solve such SDEs.

In the diffusion case one often sees drift terms as well as diffusion terms. Provided $\alpha > 1$ one can also include a drift term in the SDE (1.1) as well:

$$dX_t^i = \sum_{j=1}^d A_{ij}(X_{t-}) dZ_t^j + B_i(X_{t-}) dt.$$

This type of perturbation is easier to handle once the SDE (1.1) is solved. It can be studied using approaches in [Km3] and [TTW]. So we do not consider it in this paper.

The rest of the paper is organized as follows. Section 2 contains notation, definitions, and some facts about resolvents of certain Lévy processes. In Section 3 we derive the singular integral estimates that we need. Section 4 gives the proof of the existence of a weak solution and the connection with a martingale problem. The general idea of our proof is a Stroock-Varadhan type perturbation. However before that can be carried out, one needs an *a priori* estimate on a quantity that plays the role of the resolvent for the process X; this is done in Section 5. The perturbation argument and proof of Theorem 1.1 are in Section 6. In the last section, Section 7, weak existence and weak uniqueness for solutions to the SDE system driven by a spherically symmetric α-stable processes are studied.

2. Preliminaries.

Throughout this paper, we take the dimension $d \ge 2$. We use the letter c with or without subscripts to denote finite positive constants whose exact value is unimportant. $B(x,r)$ denotes the open ball of radius r centered at x. The inner product of a and b in $\mathbb{R}^d$ is written as $a \cdot b$. We use C_b to denote the space of bounded continuous functions on $\mathbb{R}^d$, and C_b^2 to denote the space of bounded continuous functions on $\mathbb{R}^d$ that have bounded derivatives up to second order. The notation C^∞ denote the space of continuous functions that have continuous derivatives of any order. Let C_c denote the space of functions in C_b with compact support and similarly let C_c^∞ and C_c^2 denote the collection of functions in C^∞ and C^2, respectively, with compact support. We use $\|\cdot\|_p$ to denote the L^p-norm in the space $L^p(\mathbb{R}^d, dx)$. The notation $:=$ is to be read as "is defined to be." For two real numbers a and b, $a \vee b := \max\{a, b\}$. For a function f on $\mathbb{R}^d$, its Fourier transform $\widehat{f}$ is defined by

$$\widehat{f}(u) := \int_{\mathbb{R}^d} e^{iu\cdot x} f(x)\, dx, \qquad u \in \mathbb{R}^d.$$

A one-dimensional symmetric stable process of index α is a Lévy process Z_t such that

$$\mathbb{E}\, e^{iuZ_t} = e^{-t|u|^\alpha}.$$

The Lévy measure for such a process is given by $\frac{c_\alpha}{|w|^{1+\alpha}} dw$, where c_α is a constant depending only on α.

For any process with jumps we use $X_{s-} = \lim_{t\uparrow s} X_t$ for the left hand limit and $\Delta X_s = X_s - X_{s-}$ for the jump at time s. See [M] or [E] for information about the stochastic calculus of processes with jumps. Let $\mathbb{D}$ denote the space of $\mathbb{R}^d$-valued right continuous functions on $[0, \infty)$ with left limits endowed with the Skorokhod topology.

The Lévy system formula for a one-dimensional symmetric stable process Z_t is the following. Suppose $F(x, s)(\omega)$ is a stochastic process that is jointly measurable with respect to $\mathcal{B}(\mathbb{R}) \times \mathcal{P}$ and that $\int_0^t \int_{\mathbb{R}} |F(w, s)| \frac{c_\alpha}{|w|^{1+\alpha}} dw\, ds < \infty$ for every $t > 0$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-field on $\mathbb{R}$ and $\mathcal{P}$ is the predictable σ-field generated by Z. We then have that

$$\sum_{s\le t} F(\Delta Z_s, s) - \int_0^t \int_{\mathbb{R}} F(w, s) \frac{c_\alpha}{|w|^{1+\alpha}} dw\, ds$$

is a local martingale. It will be a martingale if in addition F is bounded and $\mathbb{E} \int_0^t \int_{\mathbb{R}} |F(w, s)| \frac{c_\alpha}{|w|^{1+\alpha}} dw\, ds < \infty$ for every $t > 0$.

Assumption 2.1. *It is assumed that*

(a) *for each i, j the function $A_{ij}(x)$ is continuous and bounded;*
(b) *for each x the matrix $A(x)$ is nondegenerate, that is*

$$\mu_0(A, x) := \inf_{u\in\mathbb{R}^d : |u|=1} |A(x)u| > 0. \tag{2.1}$$

We make no symmetry assumption on A, nor do we assume that the eigenvalues of A are positive. Suppose for $j = 1, \dots, d$, that Z_t^j is a one-dimensional symmetric stable process of index α and suppose that the Z^j are independent. We consider the system of stochastic differential equations

$$X_t^i = x_0^i + \sum_{j=1}^d \int_0^t A_{ij}(X_{s-})\, dZ_s^j, \qquad i = 1, \dots, d, \tag{2.2}$$

where $x_0 = (x_0^1, \dots, x_0^d)$ is a point in $\mathbb{R}^d$.

We say that a probability measure $\mathbb{P}$ is a weak solution to the system (2.2) starting at x_0 if there exists processes $(X_t^1, \dots X_t^d, Z_t^1, \dots, Z_t^d)$ such that (2.2) holds and under $\mathbb{P}$ the processes $Z_t^1, \dots, Z_t^d$ are one-dimensional symmetric stable process of index α and the Z^j are independent. If the weak solution to (2.2) is unique, this implies in particular that the law of X is uniquely determined.

We will talk frequently about martingale problems. Let $\mathcal{M}$ be an operator whose domain includes C_b^2. Let X_t be the canonical coordinate process. We say that a probability measure $\mathbb{P}$ is a solution to the martingale problem for $\mathcal{M}$ started at $x_0 \in \mathbb{R}^d$ if $\mathbb{P}(X_0 = x_0) = 1$ and $f(X_t) - f(X_0) - \int_0^t \mathcal{M}f(X_s)ds$ is a martingale whenever $f \in C_b^2$.

From now on we will use $\mathcal{L}$ to denote the operator

$$\mathcal{L}f(x) = \sum_{j=1}^d \int_{\mathbb{R}\setminus\{0\}} \big(f(x + a_j(x)w) - f(x) - w1_{\{|w|\le 1\}} \nabla f(x) \cdot a_j(x)\big) \frac{c_\alpha}{|w|^{1+\alpha}}\, dw, \tag{2.3}$$

where $a_j(x)$ denotes the j^{th} column of the matrix $A(x)$. It will be shown in Proposition 4.1 below that $\mathcal{L}$ acts as the generator for any weak solution X of (1.1). Given a fixed point x_0 we use $\mathcal{L}_0$ for the operator

$$\mathcal{L}_0 f(x) = \sum_{j=1}^d \int_{\mathbb{R}\setminus\{0\}} \big(f(x + a_j(x_0)w) - f(x) - w1_{\{|w|\le 1\}} \nabla f(x) \cdot a_j(x_0)\big) \frac{c_\alpha}{|w|^{1+\alpha}}\, dw. \tag{2.4}$$

We will see that $\mathcal{L}_0$ is the infinitesimal generator for the process $Y_t := Y_0 + A(x_0)Z_t$. Note that Y_t is a transient Lévy process on $\mathbb{R}^d$ since Z_t is (cf. Theorem I.17 of [Be]).

Fix a point x_0 and let R_λ be the resolvent for the process Y. If we use $\mathbb{P}^x$ to denote the law of Y when $Y_0 = x$ and use $\mathbb{E}^x$ to denote its expectation, then $R_\lambda f(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda s} f(Y_s)ds$. In particular

$$R_0 f(x) = \mathbb{E}^x \int_0^\infty f(Y_s)ds. \tag{2.5}$$

Let $P_t f(x) := \mathbb{E}^x f(Y_t)$, which defines the transition semigroup of Y_t. We need the following facts about R_λ and P_t.

Proposition 2.2. *(a) For every $\lambda > 0$, there exists a positive even function r_λ on $\mathbb{R}^d$ (that is, $r_\lambda(-z) = r_\lambda(z) > 0$ for every $z \in \mathbb{R}^d$) such that if $f \in C_b$*

$$R_\lambda f(x) = \int f(y) r_\lambda(x-y)dy.$$

(b) If $1 \le p \le \infty$, then

$$\|R_\lambda f\|_p \le \|f\|_p/\lambda.$$

(c) If $p > 1$ and if we use $p(t, x-y)$ to denote the transition density function of Y with respect to the Lebesgue measure in $\mathbb{R}^d$, then

$$|P_t f(x)| \le t^{-d/(\alpha p)}\, \|p(1,\cdot)\|_q\, \|f\|_p,$$

where q is the conjugate exponent to p, that is, $\frac{1}{p} + \frac{1}{q} = 1$.

(d) Suppose $p > \max\{1, d/\alpha\}$ and $\lambda > 0$. Then

$$|R_\lambda f(x)| \le c\, \|f\|_p,$$

where $c = \|p(1, \cdot)\|_q \int_0^\infty e^{-\lambda t} t^{-d/(\alpha p)} dt$ with q being the conjugate exponent to p.

Proof. (a) Since the Lévy process Y is an affine transform of the stable process Z, Y has a positive transition density function $p(t, x-y)$ with respect to Lebesgue measure on $\mathbb{R}^d$ with $p(t, -z) = p(t, z)$ for every $z \in \mathbb{R}^d$. We then see that $r_\lambda(x-y) = \int_0^\infty e^{-\lambda t} p(t, x-y) dt$.

(b) By Jensen's inequality and the property $p(t, -z) = p(t, z)$,

$$\int_{\mathbb{R}^d} |P_t f(x)|^p \, dx \le \int_{\mathbb{R}^d} P_t(|f|^p)(x) \, dx \le \int_{\mathbb{R}^d} |f(y)|^p \, dy,$$

and so

$$\|R_\lambda f\|_p \le \int_0^\infty e^{-\lambda t} \|P_t f\|_p \, dt \le \|f\|_p / \lambda.$$

(c) By Hölder's inequality,

$$|P_t f(x)| = \left| \int_{\mathbb{R}^d} p(t, x-y) f(y) \, dy \right| \le \|f\|_p \, \|p(t, x - \cdot)\|_q = \|f\|_p \, \|p(t, \cdot)\|_q .$$

From the scaling property for the α-stable process Y, we have for every $t > 0$,

$$p(t,\, z) = t^{-d/\alpha} p(1,\, t^{-d/\alpha} z)$$

and so

$$\|p(t, \cdot)\|_q = t^{-d/\alpha}\, t^{d/(\alpha q)}\, \|p(1, \cdot)\|_q = t^{-d/(\alpha p)} \|p(1, \cdot)\|_q .$$

This proves (c).

(d) Suppose $p > \max\{1, d/\alpha\}$ and $f \in L^p(\mathbb{R}^d, dx)$. By (c),

$$\left| \int_0^\infty e^{-\lambda t} P_t f(x) dt \right| \le \|p(1, \cdot)\|_q \, \|f\|_p \int_0^\infty e^{-\lambda t} t^{-d/(\alpha p)} dt.$$

$\square$

If we calculate the Fourier transform of $\mathcal{L}_0 f$, we obtain

$$\widehat{\mathcal{L}_0 f}(u) = \widehat{f}(u) \sum_{j=1}^d \int_{\mathbb{R}\setminus\{0\}} \left(e^{iwu\cdot a_j(x_0)} - 1 - w 1_{\{|w|\le 1\}} iu \cdot a_j(x_0) \right) \frac{c_\alpha}{|w|^{1+\alpha}} \, dw.$$

Using a change of variables and the fact that $w/|w|^{1+\alpha}$ is odd, this yields

$$\widehat{\mathcal{L}_0 f}(u) = -\widehat{f}(u) \sum_{j=1}^d |u \cdot a_j(x_0)|^\alpha. \tag{2.6}$$

We conclude that

$$\widehat{R_\lambda f}(u) = \frac{\widehat{f}(u)}{\lambda + \sum_{j=1}^d |u \cdot a_j(x_0)|^\alpha}. \tag{2.7}$$

3. Singular integral estimates.

Our goal in this section is to prove some singular integral estimates. Let $\mathcal{S} = \partial B(0,1)$, let σ be surface measure on $\mathcal{S}$, and let R_0 be as in (2.5). If $a \in \mathbb{R}^d$ is nonzero, define $\mathcal{K}_a$ as an operator on $C_c^2(\mathbb{R}^d)$ by

$$\mathcal{K}_a f(x) = \int_{\mathbb{R}\setminus\{0\}} \big(R_0 f(x+ah) - R_0 f(x) - 1_{\{|h|\le 1\}} h \nabla R_0 f(x) \cdot a\big) \frac{1}{|h|^{1+\alpha}}\, dh. \tag{3.1}$$

Suppose f is C^2 with compact support. Since $Z_t = (Z_t^1, \ldots, Z_t^d)$ is transient, then $R_0 f$ is finite. By translation invariance, $R_0 f$ is C^2, and it is then easy to see that the integral in (3.1) is well defined.

In this section we will obtain a bound on the L^p-operator norm for the operator

$$f \quad \to \sup_{|a-a_0|<\delta} |\mathcal{K}_a f - \mathcal{K}_{a_0} f|, \tag{3.2}$$

where a_0 is a point in $\mathbb{R}^d \setminus \{0\}$.

Recall from (2.7) that

$$\widehat{R_0 f}(u) = \frac{1}{\sum_{j=1}^d |u \cdot a_j(x_0)|^\alpha} \widehat{f}(u). \tag{3.3}$$

A change of variables and the fact that $h/|h|^{1+\alpha}$ is odd shows that there is a real number c_2 such that

$$\int_{\mathbb{R}\setminus\{0\}} \Big(e^{-iu\cdot(ah)} - 1 + 1_{\{|h|\le 1\}}\, i\, u \cdot (ah)\Big) \frac{1}{|h|^{1+\alpha}} dh = c_2 |u \cdot a|^\alpha. \tag{3.4}$$

Combining this with (3.3), we have then that

$$\widehat{\mathcal{K}_a f}(u) = -\frac{c_3 |u \cdot a|^\alpha}{\sum_{j=1}^d |u \cdot a_j(x_0)|^\alpha} \widehat{f}(u). \tag{3.5}$$

The Fourier multiplier for $\mathcal{K}_a$ in (3.5) is bounded, and hence $\mathcal{K}_a$ is a bounded operator on L^2 by Plancherel's theorem. But the multiplier is not C^2 in $\mathbb{R}^d \setminus \{0\}$ (and if $\alpha \le 1$, it is not even C^1), so the Mihlin multiplier theorem, the Marcinkiewicz multiplier theorem, and the results of [CZ2] do not apply. We will obtain our estimates for (3.2) by using the method of rotations. We will use the following theorem, which is Theorem 2 of [CZ1]. A function f is said to be homogeneous of order k if $f(\lambda x) = \lambda^k f(x)$ for all $x \ne 0$ and $\lambda > 0$.

Theorem 3.1. (Calderon and Zygmund) *Suppose the function K is homogeneous of order $-d$,*

$$A := \int_{\mathcal{S}} |K(x)|^\gamma \sigma(dx) < \infty \tag{3.6}$$

for some $\gamma > 1$, *and*

$$\int_{\mathcal{S}} K(x)\sigma(dx) = 0.$$

Here $\mathcal{S}$ *denotes the unit sphere in* $\mathbb{R}^d$ *and* σ *is surface measure on* $\mathcal{S}$. *Then*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} K(y)1_{\{|y|>\varepsilon\}} f(x-y)dy$$

exists almost everywhere if $f \in C_c^\infty$. *If we denote the limit by* $Kf(x)$, *then for any* $p > \gamma/(\gamma-1)$, *there is a constant* c *that depends only on* d, γ *and* p *such that*

$$\|Kf\|_p \le c\, A^{1/\gamma}\, \|f\|_p \quad \text{for every } f \in C_c^\infty. \tag{3.7}$$

The following lemma will be needed. Let $Jf(x) := \int_{\mathbb{R}^d} j(x-y)f(y)dy$.

Lemma 3.2. *Suppose the kernel* $j(x)$ *of a translation invariant operator* J *is homogeneous of order* $-d$, $j(x)$ *is integrable on* $\mathcal{S}$ *with respect to* σ, *and the Fourier multiplier corresponding to the operator* J *is bounded. Then*

$$\int_{\mathcal{S}} j(x)\sigma(dx) = 0.$$

Proof. Let $B = \int_{\mathcal{S}} j(x)\sigma(dx)$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function only of $|x|$, and suppose $f \ge 0$, f is not identically 0, f is C_c^∞, and the support of f is contained in $B(0,2) \setminus B(0,1/2)$. Since $j(x)$ is homogeneous of order $-d$, we see by changing to spherical coordinates that

$$Jf(0) = \int j(x)f(0-x)dx = c_1 B,$$

and so it suffices to show $Jf(0) = 0$. Since f is C_c^∞, $Jf(x) = \int j(y)f(x-y)dy$ is continuous and $\widehat{f}$ is in the Schwartz class. Since the Fourier multiplier of J is bounded, then $\widehat{Jf} \in L^1$ and we can use the Fourier inversion formula to obtain

$$Jf(0) = (2\pi)^{-d}\int \widehat{Jf}(u)du.$$

If we denote the Fourier multiplier of J by m, then m will be homogeneous of order 0. Since f is a function only of $|x|$, then $\widehat{f}$ is a function only of $|u|$. We therefore have

$$\begin{aligned} Jf(0) &= c_2 \int m(u/|u|)\widehat{f}(|u|)du \\ &= c_3 \int_{\mathcal{S}}\int_0^\infty m(v)\widehat{f}(r)r^{d-1}dr\,\sigma(dv) \\ &= c_3\Big(\int_{\mathcal{S}} m(v)\sigma(dv)\Big)\Big(\int_0^\infty r^{d-1}\widehat{f}(r)dr\Big), \end{aligned}$$

where we write $\widehat{f}(r)$ for $\widehat{f}(u)$ if $|u| = r$. By the Fourier inversion formula,

$$\int_0^\infty r^{d-1}\widehat{f}(r)dr = c_4\int_{\mathbb{R}^d}\widehat{f}(u)du = c_5 f(0) = 0,$$

recalling that the support of f is in $B(0,2)\setminus B(0,1/2)$. We conclude that $B = 0$. □

If $q(t,x,y)$ is the transition density of a one-dimensional symmetric α-stable process, we have the following estimates.

Proposition 3.3. *There exists a constant $c_1 > 0$ such that*
(a) $q(1,0,x) \le c_1(1 \wedge |x|^{-1-\alpha})$ *and for* $k = 1,2,3,4$,

$$\frac{\partial^k}{\partial x^k} q(1,0,x) \le c_1(1 \wedge |x|^{-1-\alpha-k}).$$

(b) $q(t,0,x) \le c_1 t^{-\frac{1}{\alpha}} \left(1 \wedge (t^{\frac{1}{\alpha}}|x|^{-1})\right)^{1+\alpha}$ *and for* $k = 1,2,3,4$,

$$\frac{\partial^k}{\partial x^k} q(t,0,x) \le c_1 t^{-\frac{k+1}{\alpha}} \left((1 \wedge (t^{\frac{1}{\alpha}}|x|^{-1})\right)^{1+\alpha+k}.$$

Proof. (b) follows from (a) by scaling. For (a), the case $k = 2$ is Proposition 2.1 of [B1]. Minor modifications to the proof in [B1] give the case $k = 4$. Write

$$\frac{\partial^{k-1}}{\partial x^{k-1}} q(1,0,x) = \int_x^\infty \frac{\partial^k}{\partial x^k} q(1,0,x) dx.$$

Applying the result for $k = 4$ in this equality yields the result for $k = 3$. Repeating the argument, we then have (a) for $k = 2,1,0$. □

Let

$$\rho_t(x) = t^{-\frac{1}{\alpha}} \Big(1 \wedge \frac{t^{1+\frac{1}{\alpha}}}{|x|^{1+\alpha}}\Big). \tag{3.8}$$

Corollary 3.4. *There exists a constant c_1 such that*
(a) $q(t,0,x) \le c_1 \rho_t(x)$, $\frac{\partial}{\partial x} q(t,0,x) \le c_1 t^{-\frac{1}{\alpha}} \rho_t(x)$, *and*

$$\frac{\partial^2}{\partial x^2} q(t,0,x) \le c_1 t^{-\frac{2}{\alpha}} \rho_t(x).$$

(b) If $|x' - x| \le t^{\frac{1}{\alpha}}/2$, *then* $\rho_t(x') \le c_1 \rho_t(x)$.
(c) $\int_{\mathbb{R}} \rho_t(x) dx \le c_1$.
(d) If $\gamma > 1$, $\int_{\mathbb{R}} \rho_t(x)^\gamma dx \le c_1 t^{-\frac{\gamma-1}{\alpha}}$.

Proof. (a) follows directly from Proposition 3.3, while (b) follows from considering the two cases: $|x| \ge t^{\frac{1}{\alpha}}$ and $|x| < t^{\frac{1}{\alpha}}$ separately. (c) follows by scaling, while (d) follows from $\int \rho_t(x)^\gamma dx \le t^{-\frac{\gamma-1}{\alpha}} \int \rho_t(x) dx$. □

Let us assume for now that $A(x_0)$ is the identity. Define for $x = (x_1, x_2, \cdots, x_d)$,

$$U_\eta(x) = \int_0^\eta \prod_{i=1}^d q(t,0,x_i) dt$$

and

$$V_\eta(x) = \int_\eta^\infty \prod_{i=1}^d q(t,0,x_i) dt.$$

Clearly $R_0(x) = U_\eta(x) + V_\eta(x)$. Let

$$\mathcal{J}_a f(x) = \int_{\mathbb{R}\setminus\{0\}} \big(f(x+ah) - f(x) - 1_{\{|h|\le 1\}} h \nabla f(x)\cdot a\big) \frac{1}{|h|^{1+\alpha}}\, dh.$$

Note

$$\mathcal{K}_a f = \mathcal{J}_a R_0 f.$$

The Fourier multiplier of $\mathcal{K}_a$ is given by (3.5). Since the Fourier multiplier for $\mathcal{K}_a$ is homogeneous of order 0, it follows that $\mathcal{K}_a$ is homogeneous of order $-d$.

Proposition 3.5. *Suppose $A(x_0)$ is the identity. Let $r_0 > 0$ and suppose $a, b \in B(0, r_0)$. There exist positive constants θ_1, θ_2, and $K_1 = K_1(r_0)$ such that for each $\eta \in (0,1)$ and $x \in \mathbb{R}^d$,*

$$|(\mathcal{J}_a - \mathcal{J}_b)V_\eta(x)| \le K_1 |a-b|^{\theta_1} \eta^{-\theta_2}.$$

Proof. We look at

$$I_1 = \int_{|h|>1} |V_\eta(x+ah) - V_\eta(x+bh)| \frac{1}{|h|^{1+\alpha}} dh,$$
$$I_2 = \int_{|h|\le 1} \Big|V_\eta(x+ah) - V_\eta(x+bh) - h\nabla V_\eta(x)\cdot(a-b)\Big| \frac{1}{|h|^{1+\alpha}}\, dh.$$

From Proposition 3.3(b) we see that

$$V_\eta(x) \le c_1 \int_\eta^\infty t^{-d/\alpha} dt \le c_2 \eta^{1-\frac{d}{\alpha}}.$$

Note that $d \ge 2 > \alpha$. Similarly,

$$|\nabla V_\eta(x)| \le c_3 \eta^{1-\frac{d+1}{\alpha}}. \tag{3.9}$$

If $|a-b|\,|h| > 1$, then

$$|V_\eta(x+ah) - V_\eta(x+bh)| \le 2\|V_\eta\|_\infty \le 2\|V_\eta\|_\infty |h|^{\alpha/2} |a-b|^{\alpha/2}.$$

If $|a-b|\,|h| \le 1$, then

$$|V_\eta(x+ah) - V_\eta(x+bh)| \le \|\nabla V_\eta\|_\infty |a-b|\,|h| \le \|\nabla V_\eta\|_\infty |a-b|^{\alpha/2} |h|^{\alpha/2}.$$

Substituting in the formula for I_1,

$$I_1 \le c_4 \big(\|V_\eta\|_\infty + \|\nabla V_\eta\|_\infty\big) |a-b|^{\alpha/2} \int_{|h|>1} \frac{|h|^{\alpha/2}}{|h|^{1+\alpha}}\, dh \le c_5 \eta^{-\beta_1} |a-b|^{\alpha/2}$$

for some $\beta_1 > 0$.

We turn to I_2. Applying a Taylor series expansion to the function $h \mapsto V_\eta(x+ah) - V_\eta(x+bh)$, we have

$$I_2 \le \int_{|h|\le 1} |h|^2 \sup_{i,j} \left\| \frac{\partial^2 V_\eta}{\partial x_i \partial x_j}(y+ah) - \frac{\partial^2 V_\eta}{\partial x_i \partial x_j}(y+bh) \right\|_\infty \frac{1}{|h|^{1+\alpha}} dh.$$

From Proposition 3.3(b), we obtain the estimate

$$\sup_{i,j,k} \left\| \frac{\partial^3 V_\eta}{\partial x_i \partial x_j \partial x_k}(x) \right\|_\infty \le c_6 \eta^{1-\frac{d+3}{\alpha}},$$

and so I_2 is bounded above by

$$I_3 \le c_7 \sup_{i,j,k} \left\| \frac{\partial^3 V_\eta}{\partial x_i \partial x_j \partial x_k}(y) \right\|_\infty |a-b| \le c_8 \eta^{-\beta_2} |a-b|.$$

Combining our bounds for I_1 and I_2 proves the proposition. □

In order to estimate $\mathcal{J}_a U_\eta$, we first need the following lemma. Let

$$P_t(x) = \prod_{i=1}^d q(t, 0, x_i). \tag{3.10}$$

Lemma 3.6. *For $\lambda > 0$, $U_\eta(\lambda x) = \lambda^{\alpha-d} U_{\eta/\lambda^\alpha}(x)$.*

Proof. By scaling, $q(t, 0, \lambda x) = \lambda^{-1} q(t/\lambda^\alpha, 0, x)$. So

$$\begin{aligned} U_\eta(\lambda x) &= \int_0^\eta \prod_{i=1}^d q(t, 0, \lambda x_i) dt \\ &= \lambda^{-d} \int_0^\eta \prod_{i=1}^d q(t/\lambda^\alpha, 0, x_i) dt, \end{aligned}$$

and our result follows by a change of variable. □

Proposition 3.7. *Suppose $A(x_0)$ is the identity. Let $r_0 > 1 > r_1 > 0$ and suppose $a \in B(0, r_0) \setminus B(0, r_1)$. There exist positive constants $\gamma > 1, \theta_3$, and $K_2 = K_2(r_0, r_1)$ such that for each $\eta \in (0, 1/\sqrt{d})$*

$$\int_{\mathcal{S}} |\mathcal{J}_a U_\eta(x)|^\gamma \sigma(dx) \le K_2 \eta^{\theta_3}.$$

Proof.

Step 1. We make an observation that will be used later. Note that since $h/|h|^{1+\alpha}$ is an odd function,

$$\mathcal{J}_a f(x) = \int_{\mathbb{R}\setminus\{0\}} [f(x+ah) - f(x) - 1_{\{|h| \le \psi(x)\}} h \nabla f(x) \cdot a] \frac{1}{|h|^{1+\alpha}} dh$$

whenever $\psi(x) \in (0, \infty)$.

Step 2. Let $N = 2(r_0+1)\sqrt{d}$. In this step we want to show there exists $\gamma > 1$ such that

$$\int_{\mathcal{S}} \int_{|h| \ge 1/N} |U_\eta(x)|^\gamma \frac{1}{|h|^{1+\alpha}} dh \, \sigma(dx) \le c_{21} \eta^{\beta_1}. \tag{3.11}$$

Let $F \geq 0$ be in $L^p(\mathcal{S}, \sigma)$ with $\|F\|_p \leq 1$, where p will be chosen later. Extend the definition of F to $\mathbb{R}^d \setminus \{0\}$ by setting $F(x) = F(x/|x|)$. Let $f \geq 0$ be a continuous function with support in $B(0, 2^{1/\alpha}) \backslash B(0,1)$ and taking values in $[0,1]$ such that $f(x)$ depends only on $|x|$ and $\int_{\mathbb{R}^d} f(x)dx > 0$. Using Lemma 3.6, if $2^{-1/\alpha} < \lambda < 1$, then $U_\eta(x) \leq 2^{\frac{d}{\alpha}-1} U_{2\eta}(\lambda^{-1}x)$. So

$$\begin{aligned}\int_{\mathcal{S}} U_\eta(x)F(x)\sigma(dx) &\leq c_{22} \int U_{2\eta}(x)F(x)f(x)dx \\ &= c_{22} \int_0^{2\eta} P_t(fF)(0)dt.\end{aligned}$$

By Proposition 2.2(c), for $p > 1$,

$$P_t(fF)(0) \leq c_{23} t^{-d/(\alpha p)},$$

where the constant $c_{23} > 0$ can be chosen to depend only on d and p. Therefore if $p > \max\{1,\ d/\alpha\}$,

$$\int_{\mathcal{S}} U_\eta(x)F(x)\sigma(dx) \leq c_{24}\eta^{\beta_2}.$$

So for $\gamma = p/(p-1)$, taking the supremum over such F and using duality yields

$$\int_{\mathcal{S}} |U_\eta(x)|^\gamma \sigma(dx) \leq c_{24}\eta^{\beta_2}. \tag{3.12}$$

Since $\int_{|h|>1/N} |h|^{-1-\alpha} dh$ is finite, (3.11) follows.

Step 3. Let $\mathcal{S}^+ = \{x \in \mathcal{S} : a \cdot x > \frac{3}{4}|a|\}$ and $\mathcal{S}^- = \{x \in \mathcal{S} : -x \in \mathcal{S}^+\}$. In this step we will show there exists $\gamma > 1$ such that

$$\int_{\mathcal{S}^+} \left| \int_{|h|>1/N} U_\eta(x+ah) \frac{1}{|h|^{1+\alpha}} dh \right|^\gamma \sigma(dx) \leq c_{31}\eta^{\beta_4} \tag{3.13}$$

and

$$\int_{\mathcal{S}^-} \left| \int_{|h|>1/N} U_\eta(x+ah) \frac{1}{|h|^{1+\alpha}} dh \right|^\gamma \sigma(dx) \leq c_{32}\eta^{\beta_5}, \tag{3.14}$$

where N is the constant given in Step 2. The two cases are similar so we will only look at $\mathcal{S}^+$. Let F be a bounded function on $\mathcal{S}^+$ with $\|F\|_{L^p(\mathcal{S}^+,\sigma)} \leq 1$ and extend the definition of F to $\{x + h\,a :\ x \in \mathcal{S}^+ \text{ and } h \in \mathbb{R}\}$ by setting $F(x + ah) = F(x)$. Let k be an integer other than 0 or -1. It is easy to see that $dh\,\sigma(dx)$ has a Radon-Nikodym derivative with respect to the Lebesgue measure on $D_k := \{x + ah : x \in \mathcal{S}^+, k/N \leq h < (k+1)/N\}$ that is bounded above and below by positive constants. We write

$$\begin{aligned}&\int_{\mathcal{S}^+} \int_{k/N}^{(k+1)/N} U_\eta(x+ah)F(x)\frac{1}{|h|^{1+\alpha}} dh\,\sigma(dx) \\ &\qquad = \int_{\mathcal{S}^+} \int_{k/N}^{(k+1)/N} U_\eta(x+ah)F(x+ah)\frac{1}{|h|^{1+\alpha}} dh\,\sigma(dx) \\ &\qquad \leq c_{33}|k|^{-(1+\alpha)} \int_{\mathcal{S}^+} \int_{k/N}^{(k+1)/N} U_\eta(x+ah)F(x+ah) dh\,\sigma(dx) \\ &\qquad \leq c_{34}|k|^{-(1+\alpha)} \int_{D_k} U_\eta(y)F(y)\,dy \\ &\qquad = c_{34}|k|^{-(1+\alpha)} \int_0^\eta P_t(F1_{D_k})(0)dt.\end{aligned}$$

Using Proposition 2.2(c) and by choosing p large enough so that $\beta_6 = 1 - \frac{d}{\alpha(p-1)} > 0$, the above is bounded by

$$c_{35}|k|^{-(1+\alpha)}\eta^{\beta_6}.$$

Summing over k,

$$\int_{\mathcal{S}^+}\int_{|h|>1/N} U_\eta(x+ah)|F(x)|\frac{1}{|h|^{1+\alpha}}dh\,\sigma(dx) \le c_{36}\eta^{\beta_6},$$

and taking the supremum over such F and using duality, (3.13) follows with $\gamma = p/(p-1)$.

Step 4. Let $\mathcal{S}_0^+ = \{x \in \mathcal{S} : \frac{3}{4}|a| \ge a\cdot x \ge 0\}$ and $\mathcal{S}_0^- = \{x \in \mathcal{S} : -x \in \mathcal{S}_0^+\}$. In this step we prove there exists $\gamma > 1$ such that

$$\int_{\mathcal{S}_0^+}\int_{|h|>1/N} |U_\eta(x+ah)|^\gamma\frac{1}{|h|^{1+\alpha}}dh\,\sigma(dx) \le c_{41}\eta^{\beta_7} \tag{3.15}$$

and we then use the same argument for $\mathcal{S}_0^-$.

We first consider

$$\int_{\mathcal{S}_0^+} |U_\eta(x+ah)|^\gamma\sigma(dx) \tag{3.16}$$

for $|h| \ge 1/N$. Let σ_h be the translation of σ by ah, and let μ_h be the projection of σ_h onto $\mathcal{S}$. To be more precise, $\sigma_h(A) = \sigma(A - ah)$ if $A \subset \mathbb{R}^d$ and $\mu_h(B) = \sigma_h(\{y : y/|y| \in B\})$ if $B \subset \mathcal{S}$.

We claim that the Radon-Nikodym derivative of μ_h with respect to σ is bounded by $c_{42}(1+h^2)^{d/2}$ for $|h| \ge 1/N$. This can be seen by a computation. Another way to see this is the following. Let e_i be the unit vector in the i^{th} coordinate direction, and suppose for the moment that $a = e_d$. Then a point $x \in \mathcal{S}_0^+$ gets mapped to $y_0 = x + he_d$ and $x + \varepsilon e_d$ gets mapped to $y_d = x + (\varepsilon + h)e_d$. Since $x \in \mathcal{S}_0^+$, by the mean value theorem the distance between the projections of y_0 and y_d onto $\mathcal{S}$ is no less than

$$c_{43}\left|\arctan\left(\frac{x_d + h}{\sqrt{1-x_d^2}}\right) - \arctan\left(\frac{x_d + h + \varepsilon}{\sqrt{1-x_d^2}}\right)\right| \ge c_{44}\frac{\varepsilon}{1+h^2}.$$

A similar argument shows that if v is a unit vector tangential to $\mathcal{S}_0^+$ at x and orthogonal to e_d, then $x+\varepsilon v$ gets mapped to $y_v = x+\varepsilon v+he_d$, and in this case the distance between the projections of y_0 and y_v onto $\mathcal{S}$ is bounded below by $c_{45}\varepsilon/\sqrt{1+h^2}$. Since the tangent space to $\mathcal{S}_0^+$ is $(d-1)$-dimensional, the claimed bound on the Radon-Nikodym derivative follows. By a rotation, the same estimate holds for any other a.

We now use Lemma 3.6 and write

$$\begin{aligned}\int_{\mathcal{S}_0^+} |U_\eta(x+ah)|^\gamma\sigma(dx) &= \int_{\mathcal{S}_0^+ + ah} |U_\eta(x)|^\gamma\sigma_h(dx)\\ &\le c_{46}|h|^{(\alpha-d)\gamma}\int_{\mathcal{S}} |U_{\eta/|h|^\alpha}(x)|^\gamma\mu_h(dx)\\ &\le c_{47}|h|^{(\alpha-d)\gamma}(1+h^2)^{d/2}\int_{\mathcal{S}} |U_{\eta/|h|^\alpha}(x)|^\gamma\sigma(dx).\end{aligned}$$

By (3.12) the L^γ-norm of $U_{\eta/|h|^\alpha}$ is bounded by $c_{48}(\eta/|h|^\alpha)^{\beta_3/\gamma}$, and so

$$\int_{\mathcal{S}_0^+} |U_\eta(x+ah)|^\gamma \sigma(dx) \le c_{47}\, c_{48}\, |h|^{(\alpha-d)\gamma-\beta_3\alpha}\, (1+h^2)^{d/2}\, \eta^{\beta_3}.$$

Provided $\gamma > 1$ is chosen sufficiently close to 1, we can now multiply both sides of this inequality by $|h|^{-1-\alpha}$ and integrate over $|h| \ge 1/N$ to obtain (3.15).

Step 5. Combining Steps 2, 3, and 4 and using Hölder's inequality, we see that there are positive constants $\gamma > 1$ and $\beta_8 > 0$ such that

$$\int_{\mathcal{S}} \Big| \int_{|h|>1/N} |U_\eta(x+ah) - U_\eta(x)| \frac{1}{|h|^{1+\alpha}} dh \Big|^\gamma \sigma(dx) \le c_{51}\eta^{\beta_8}. \tag{3.17}$$

Step 6. Let $\mathcal{S}_1 = \{x \in \mathcal{S} : 0 < x_1 < x_2 < \cdots < x_d\}$. For $x \in \mathcal{S}_1$ note that since $\sum_{i=1}^d x_i^2 = 1$, then $x_d \ge 1/\sqrt{d}$ and $x_{d-1} \le 1/\sqrt{2}$. We will show that there are constants $\gamma > 1$ and $\zeta_1 > -1$ such that

$$\int_{\mathcal{S}_1} \Big| \int_{|h|\le t^{\frac{1}{\alpha}}/N} |P_t(x+ah) - P_t(x) - h\nabla P_t(x)\cdot a| \frac{1}{|h|^{1+\alpha}} dh \Big|^\gamma \sigma(dx) \le c_{61} t^{\zeta_1}, \tag{3.18}$$

where the function $P_t(x)$ is given by (3.10). By a Taylor series expansion,

$$|P_t(x+ah) - P_t(x) - h\nabla P_t(x)\cdot a| \le c_{62} h^2 \sup_{|s|\le h} \sup_{k,\ell} \Big| \frac{\partial^2 P_t}{\partial x_k \partial x_\ell}(x+as) \Big|.$$

Since

$$\int_{|h|\le t^{\frac{1}{\alpha}}/N} |h|^2 \frac{1}{|h|^{1+\alpha}} dh \le c_{63} |t^{\frac{1}{\alpha}}|^{2-\alpha},$$

using Corollary 3.4(a) and (b) the left hand side of (3.18) is bounded by

$$c_{64} \int_{\mathcal{S}_1} \Big| |t^{\frac{1}{\alpha}}|^{2-\alpha} t^{-\frac{2}{\alpha}} \prod_{i=1}^d \rho_t(x_i) \Big|^\gamma \sigma(dx).$$

Because $x_d \ge 1/\sqrt{d}$, $\rho_t(x_d) \le c_{65} t$. Therefore the above is bounded by

$$c_{66} \int_{\mathcal{S}_1} \prod_{i=1}^{d-1} \rho_t(x_i)^\gamma \sigma(dx).$$

Recall $x_1 < x_2 < \cdots < x_{d-1} \le 1/\sqrt{2}$ on $\mathcal{S}_1$. When restricted to $\mathcal{S}_1$, the Radon-Nikodym derivative of $\sigma(dx)$ on $\mathcal{S}$ with respect to Lebesgue measure in $\mathbb{R}^{d-1}$ restricted to $\{0 < x_1 < \cdots < x_{d-1} < 1/\sqrt{2}\}$ is bounded between two positive constants. We integrate over $x_1, \ldots, x_{d-1}$ and use Corollary 3.4(d). We see that the left hand side of (3.18) is bounded by $c_{67} t^{-(\gamma-1)(d-1)/\alpha}$. By taking $\gamma > 1$ sufficiently close to 1, the exponent will be larger than -1. This establishes (3.18).

Step 7. Let $x \in \mathcal{S}_1$. We will show that there are constants $\gamma > 1$ and $\zeta_2 > -1$ such that for $t \le 1$,

$$\int_{\mathcal{S}_1} \Big| \int_{t^{\frac{1}{\alpha}}/N<|h|\le 1/N} (P_t(x+ah) - P_t(x)) \frac{1}{|h|^{1+\alpha}} dh \Big|^\gamma \sigma(dx) \le c_{71} t^{\zeta_2}. \tag{3.19}$$

Suppose $F \in L^p(\mathcal{S}_1, \sigma)$ with norm less than or equal to 1, where $p^{-1} + \gamma^{-1} = 1$ and γ will be chosen later. By Hölder's inequality,

$$\begin{aligned}\Big|\int_{\mathcal{S}_1} \int_{t^{\frac{1}{\alpha}}/N<|h|\le 1/N} &[P_t(x+ah) - P_t(x)] \frac{1}{|h|^{1+\alpha}} dh\, F(x)\sigma(dx)\Big| \\ &\le \int_{t^{\frac{1}{\alpha}}/N<|h|\le 1/N} \Big(\int_{\mathcal{S}_1} |P_t(x+ah) - P_t(x)|\, |F(x)|\sigma(dx)\Big) \frac{1}{|h|^{1+\alpha}} dh \\ &\le \int_{t^{\frac{1}{\alpha}}/N<|h|\le 1/N} \Big(\int_{\mathcal{S}_1} |P_t(x+ah) - P_t(x)|^\gamma\; \sigma(dx)\Big)^{1/\gamma} \frac{1}{|h|^{1+\alpha}} dh.\end{aligned} \tag{3.20}$$

Since $|h| \le 1/N$, then $|x_d + a_d h| \ge 1/(2\sqrt{d})$, so $\rho_t(x_d + a_d h) \le c_{72} t$. Therefore

$$P_t(x+ah) \le c_{73} t \prod_{i=1}^{d-1} \rho_t(x_i + a_i h).$$

As in Step 6, integration gives

$$\int t^\gamma \prod_{i=1}^{d-1} \rho_t(x_i + a_i h)^\gamma \sigma(dx) \le c_{74} t^{\gamma - \frac{(\gamma-1)(d-1)}{\alpha}}.$$

We have a similar estimate for $\int P_t(x)^\gamma \sigma(dx)$. We also have

$$\int_{t^{\frac{1}{\alpha}}/N<|h|} \frac{1}{|h|^{1+\alpha}} dh \le c_{75} t^{-1},$$

and so the left hand side of (3.20) is bounded by

$$c_{76} t^{-(\gamma-1)(d-1)/(\gamma\alpha)}.$$

Taking the supremum over such F, using duality, and taking $\gamma > 1$ sufficiently close to 1, we have (3.19).

Step 8. In this step we will show that there are constants $\gamma > 1$ and $\beta_9 > 0$ such that

$$\int_{\mathcal{S}_1} \Big|\int_{|h|\le 1/N} |U_\eta(x+ah) - U_\eta(x) - h\nabla U_\eta(x)\cdot a| \frac{1}{|h|^{1+\alpha}} dh\Big|^\gamma \sigma(dx) \le c_{81} \eta^{\beta_9}. \tag{3.21}$$

Suppose $F \in L^p(\mathcal{S}_1, \sigma)$ with norm less than or equal to 1, where $p^{-1} + \gamma^{-1} = 1$. Then by Steps 1, 6 and 7,

$$
\begin{aligned}
&\Big|\int_{\mathcal{S}_1}\Big(\int_{|h|\le 1/N}\big|U_\eta(x+ah)-U_\eta(x)-h\nabla U_\eta(x)\cdot a\big|\frac{1}{|h|^{1+\alpha}}dh\Big)F(x)\,\sigma(dx)\Big|\\
&=\Big|\int_{\mathcal{S}_1}\Big(\int_{|h|\le 1/N}|U_\eta(x+ah)-U_\eta(x)-h1_{\{|h|\le t^{1/\alpha}/N\}}\nabla U_\eta(x)\cdot a|\,\frac{1}{|h|^{1+\alpha}}dh\Big)F(x)\,\sigma(dx)\Big|\\
&\le\int_{\mathcal{S}_1}\Big(\int_{|h|\le 1/N}\Big(\int_0^\eta\big|P_t(x+ah)-P_t(x)-h1_{\{|h|\le t^{1/\alpha}/N\}}\nabla P_t(x)\cdot a\big|\,dt\Big)\frac{dh}{|h|^{1+\alpha}}\Big)\\
&\qquad\times|F(x)|\,\sigma(dx)\\
&=\int_0^\eta\int_{\mathcal{S}_1}\Big(\int_{|h|\le 1/N}\big|P_t(x+ah)-P_t(x)-h1_{\{|h|\le t^{1/\alpha}/N\}}\nabla P_t(x)\cdot a\big|\,\frac{dh}{|h|^{1+\alpha}}\Big)\,|F(x)|\,\sigma(dx)\,dt\\
&\le\Big(\int_0^\eta\int_{\mathcal{S}_1}\Big|\int_{|h|\le 1/N}\big|P_t(x+ah)-P_t(x)-h1_{\{|h|\le t^{1/\alpha}/N\}}\nabla P_t(x)\cdot a\big|\,\frac{dh}{|h|^{1+\alpha}}\Big|^\gamma\sigma(dx)\,dt\Big)^{1/\gamma}\\
&\qquad\times\Big(\int_0^\eta\int_{\mathcal{S}_1}|F(x)|^p\sigma(dx)\,dt\Big)^{1/p}\\
&\le c_{82}\Big(\int_0^\eta[t^{\zeta_1}+t^{\zeta_2}]dt\Big)^{1/\gamma}\eta^{1/p}.
\end{aligned}
$$

Taking the supremum over such F and using duality, (3.21) follows.

Step 9. By symmetry, we have the same estimate as (3.21) over $\mathcal{S}_\pi=\{x\in\mathcal{S}:0<x_{\pi(1)}<\cdots<x_{\pi(d)}\}$, where $(\pi(1),\dots,\pi(d))$ is a permutation of the indices. Since the set of points in the orthant $\mathcal{O}=\{x:x_1,\dots,x_d\ge 0\}$ that are not in $\mathcal{S}_\pi$ for any permutation π has σ-measure 0, we conclude

$$
\int_{\mathcal{O}}\Big|\int_{|h|\le 1/N}|U_\eta(x+ah)-U_\eta(x)-h\nabla U_\eta(x)\cdot a|\frac{1}{|h|^{1+\alpha}}dh\Big|^\gamma\sigma(dx)\le c_{91}\eta^{\beta_9}. \tag{3.22}
$$

Using symmetry, we have the same estimate over each of the 2^d orthants. Combining these estimates we obtain

$$
\int_{\mathcal{S}}\Big|\int_{|h|\le 1/N}|U_\eta(x+ah)-U_\eta(x)-h\nabla U_\eta(x)\cdot a|\frac{1}{|h|^{1+\alpha}}dh\Big|^\gamma\sigma(dx)\le c_{92}\eta^{\beta_9}. \tag{3.23}
$$

Estimates (3.17) and (3.23) together prove the proposition. □

Corollary 3.8. *Suppose $A(x_0)$ is the identity matrix. Suppose $r_0>1>r_1>0$ and $a,b\in B(0,r_0)\setminus B(0,r_1)$. There exist positive constants $K_3=K_3(r_0,r_1)$ and θ such that*

$$
\int_{\mathcal{S}}|(\mathcal{J}_a-\mathcal{J}_b)R_0(x)|^\gamma\sigma(dx)\le K_3|a-b|^\theta,
$$

where γ is the constant in Proposition 3.7.

Proof. By Proposition 3.5,

$$
\int_{\mathcal{S}}|(\mathcal{J}_a-\mathcal{J}_b)V_\eta(x)|^\gamma\sigma(dx)\le c_1K_1^\gamma|a-b|^{\gamma\theta_1}\eta^{-\gamma\theta_2}.
$$

Since $R_0 = U_\eta + V_\eta$, this and Proposition 3.7 imply

$$\int_{\mathcal{S}} |(\mathcal{J}_a - \mathcal{J}_b)R_0(x)|^\gamma \sigma(dx) \le c_2 K_1^\gamma |a-b|^{\gamma\theta_1} \eta^{-\gamma\theta_2} + c_2 K_2 \eta^{\theta_3}.$$

If we now select

$$\eta = \Big(\frac{K_1^\gamma |a-b|^{\gamma\theta_1}}{K_2} \Big)^{1/(\gamma\theta_2+\theta_3)} \wedge 1,$$

we obtain our result with $\theta = \gamma\theta_1\theta_3/(\gamma\theta_2+\theta_3)$. □

Corollary 3.9. *Let γ be the constant in Proposition 3.7, let $r_0 > 1 > r_1 > 0$, $p > \gamma/(\gamma-1)$, and $a, b \in B(0,r_0) \setminus B(0,r_1)$. There exist positive constants $K_4 = K_4(r_0, r_1, \gamma, p)$ and θ_4 such that*

$$\|(\mathcal{K}_a - \mathcal{K}_b)f\|_p \le K_4 |a-b|^{\theta_4} \|f\|_p.$$

Proof. By Corollary 3.8, $(\mathcal{J}_a - \mathcal{J}_b)R_0$ is integrable over $\mathcal{S}$. By the discussion preceding Proposition 3.5, it is homogeneous of order $-d$. So by Lemma 3.2 its integral over $\mathcal{S}$ is 0. Our result now follows by Corollary 3.8 and Theorem 3.1. □

At this point we could follow the argument in [CZ2] and obtain a bound on operators of the form

$$(\mathcal{K}_{a(x)} - \mathcal{K}_{a_0})f(x),$$

but we actually need to be able to handle certain random operators. So we choose to proceed as follows.

Remark. We will see from its proof that Theorem 3.10 is a fairly general result which holds as long as the conclusion of Corollary 3.9 is valid and does not use specific information about $\mathcal{K}_a$.

Theorem 3.10. *Suppose $A(x_0)$ is the identity matrix. Let $r_0 > 1$ and γ, θ_4 be as in Corollary 3.9. Fix $a_0 \in B(0,r_0) \setminus \{0\}$ and let $\delta \in (0, 1 \wedge (|a_0|/2))$. Define*

$$\overline{p} := \Big(\frac{\gamma}{\gamma-1} \Big) \vee \Big(\frac{2d}{\theta_4} \Big) \vee \Big(\frac{d}{\alpha} \Big). \tag{3.24}$$

Then for every $p > \overline{p}$, there exist a constant $K_5 = K_5(r_0, p, \gamma) > 0$ and a constant $\theta_5 = \theta_5(p) > 0$ such that

$$\Big\| \sup_{a \in B(0,r_0), |a-a_0|<\delta} |\mathcal{K}_a f(x) - \mathcal{K}_{a_0} f(x)| \Big\|_p \le K_5 \delta^{\theta_5} \|f\|_p. \tag{3.25}$$

Proof. Let $\mathcal{A}_m$ be a 2^{-m}-net for $B(0,r_0)$ with respect to the Euclidean topology such that $\mathcal{A}_{m+1} \supset \mathcal{A}_m$ and the cardinality of $\mathcal{A}_m$ is less than $c_1 2^{md}$. It is routine to construct such $\mathcal{A}_m$'s. Let $B = B(0,r_0) \cap B(a_0, \delta)$. Clearly it suffices to establish (3.25) for $f \in C_c^\infty$. By an easy continuity argument it is enough to show

$$\Bigg\| \sup_{a \in B \cap (\cup_m \mathcal{A}_m)} |\mathcal{K}_a f(x) - \mathcal{K}_{a_0} f(x)| \Bigg\|_p \le K_5 \delta^{\theta_5} \|f\|_p$$

By Fatou's lemma, it is enough to show

$$\Big\| \sup_{a\in B\cap\mathcal{A}_m} |\mathcal{K}_a f(x) - \mathcal{K}_{a_0} f(x)| \Big\|_p \le K_5 \delta^{\theta_5} \|f\|_p \tag{3.26}$$

with K_5 independent of m.

Choose k such that $\delta \le 2^{-k} < 2\delta$. Let b_j be the element of $\mathcal{A}_j$ closest to a, with some convention for breaking ties. Let $\mathcal{H}_a f(x) = \mathcal{K}_a f(x) - \mathcal{K}_{a_0} f(x)$ and for $a \in \mathcal{A}_m$ write

$$\mathcal{H}_a f(x) = \mathcal{H}_{b_k} f(x) + (\mathcal{H}_{b_{k+1}} f(x) - \mathcal{H}_{b_k} f(x)) + (\mathcal{H}_{b_{k+2}} f(x) - \mathcal{H}_{b_{k+1}} f(x)) + \cdots. \tag{3.27}$$

For $a \in \mathcal{A}_m$ this is actually a finite sum.

We claim that if $\sup_{a\in\mathcal{A}_m\cap B} |\mathcal{H}_a f(x)|$ is larger than λ, then either

(a) $|\mathcal{H}_a f(x)|$ is larger than $\lambda/2$ for some $a \in B \cap \mathcal{A}_k$

or

(b) for some $j \ge k$, there exist $b_1 \in \mathcal{A}_j$ and $b_2 \in \mathcal{A}_{j+1}$ with $|b_1 - b_2| \le 2^{-j+1}$ and

$$|\mathcal{K}_{b_1} f(x) - \mathcal{K}_{b_2} f(x)| > \lambda/(6j^2).$$

For if not, then by (3.27)

$$|\mathcal{H}_a f(x)| \le \frac{\lambda}{2} + \sum_{j=k}^{\infty} \frac{\lambda}{6j^2} < \lambda.$$

Let

$$C_k = \{x \in \mathcal{S} : \exists\, b \in \mathcal{A}_k \cap B \text{ such that } |\mathcal{H}_b f(x)| > \lambda/2\}.$$

If $b \in \mathcal{A}_k \cap B$, Chebyshev's inequality and Corollary 3.9 tell us that

$$\begin{aligned}
\sigma\{x \in \mathcal{S} : |\mathcal{H}_b f(x)| > \lambda/2\} &\le \frac{\|\mathcal{H}_b f\|_p^p}{(\lambda/2)^p} \\
&\le \frac{\|\mathcal{J}_b R_0 f - \mathcal{J}_{a_0} R_0 f\|_p^p}{(\lambda/2)^p} \\
&\le c_2 \frac{|b - a_0|^{p\theta_4} \|f\|_p^p}{\lambda^p} \\
&\le c_2 \delta^{p\theta_4} \frac{\|f\|_p^p}{\lambda^p}.
\end{aligned}$$

Since the cardinality of $\mathcal{A}_k$ is bounded by $c_1 2^{kd}$, we have

$$\sigma(C_k) \le c_1 2^{kd} c_2 \delta^{p\theta_4} \frac{\|f\|_p^p}{\lambda^p} \le c_3 \delta^{p\theta_4 - d} \|f\|_p^p / \lambda^p. \tag{3.28}$$

Let

$$D_j = \{x \in \mathcal{S} : \exists\, b_1, b_2 \in \mathcal{A}_{j+1} \text{ such that } |b_1 - b_2| \le 2^{-j+1}, |\mathcal{H}_{b_1} f(x) - \mathcal{H}_{b_2} f(x)| > \lambda/(6j^2)\}.$$

Chebyshev's inequality and Corollary 3.9 tell us that

$$\begin{aligned}\sigma\{x \in \mathcal{S} : |\mathcal{H}_{b_1} f(x) - \mathcal{H}_{b_2} f(x)| > \lambda/6j^2\} &\leq \frac{\|\mathcal{H}_{b_1} f - \mathcal{H}_{b_2} f\|_p^p}{(\lambda/6j^2)^p} \\ &= \frac{\|\mathcal{J}_{b_1} R_0 f - \mathcal{J}_{b_2} R_0 f\|_p^p}{(\lambda/6j^2)^p} \\ &\leq c_4 \frac{|b_1 - b_2|^{p\theta_4} j^{2p} \|f\|_p^p}{\lambda^p}.\end{aligned}$$

Since the cardinality of the set of such pairs b_1, b_2 in $\mathcal{A}_{j+1}$ is bounded by $c_1^2 2^{(2j+2)d}$, we obtain

$$\sigma(D_j) \leq c_5 2^{2jd} j^{2p} 2^{-jp\theta_4} \frac{\|f\|_p^p}{\lambda^p}. \tag{3.29}$$

Let

$$E = C_k \bigcup \Big(\cup_{j=k}^{\infty} D_j \Big).$$

As long as $p > 2d/\theta_4$, then $\sum_{j=k}^{\infty} \sigma(D_j)$ converges, and there exist positive constants θ_5 and c_6 such that

$$\sigma(E) \leq c_6 \delta^{p\theta_5} \|f\|_p^p / \lambda^p. \tag{3.30}$$

But by our argument above, E contains all points $x \in \mathcal{S}$ such that $\sup_{a \in B \cap \mathcal{A}_m} |\mathcal{H}_a f(x)|$ is larger than λ. In other words, we have a weak (p,p) inequality for the operator

$$f \to \sup_{a \in B \cap \mathcal{A}_m} |\mathcal{K}_a f(x) - \mathcal{K}_{a_0} f(x)|$$

for any $p > \overline{p}$. Our result now follows by the Marcinkiewicz interpolation theorem (see, for example, Appendix B in Stein [S]). □

It remains to remove the restriction that $A(x_0)$ is the identity. So suppose $A(x_0)$ is nondegenerate, let G be the inverse of $A(x_0)$ and let

$$N = \sum_{i,j} (|A_{ij}(x_0)| + |G_{ij}|).$$

Define

$$\mathcal{G}_{a_0,\delta} f(x) = \sup_{a \in B(0,r_0), |a-a_0|<\delta} |\mathcal{K}_a f(x) - \mathcal{K}_{a_0} f(x)|. \tag{3.31}$$

Theorem 3.11. *Let $r_0 > 1$, $0 < \delta < 1 \wedge |a_0|/2$, and let $\overline{p} > 1$ be the constant in Theorem 3.10. Then for every $p > \overline{p}$, there exists a positive constant $K_6 = K_6(r_0, N, \gamma, p)$ such that*

$$\|\mathcal{G}_{a_0,\delta} f\|_p \leq K_6 \delta^{\theta_5} \|f\|_p. \tag{3.32}$$

Proof. Let $\widetilde{f}(x) = f(A(x_0)x)$. Let R_0 be the potential corresponding to $A(x_0)Z_t$ and S_0 be the potential corresponding to Z_t. Note

$$R_0 f(x) = \mathbb{E}^x \int_0^{\infty} f(A(x_0) Z_s) ds = \mathbb{E}^x \int_0^{\infty} \widetilde{f}(Z_s) ds = S_0 \widetilde{f}(x).$$

Let

$$\mathcal{G}f(x) = \sup_{a \in B(0,r_0), |a-a_0|<\delta} |\mathcal{J}_a f(x) - \mathcal{J}_{a_0} f(x)|.$$

By Theorem 3.10,

$$\|\mathcal{G}R_0 f\|_p = \|\mathcal{G}S_0 \widetilde{f}\|_p \le c_1 \delta^{\theta_5} \|\widetilde{f}\|_p.$$

Clearly the norms of f and $\widetilde{f}$ are comparable (with a constant depending on N), and our result follows. □

Remark. From Proposition 3.3 we see that $\prod_{i=1}^d q(t, x_i, y_i) \le c_1 t^{-d/\alpha}$. It is well known that this implies that Z_t is transient. It follows that $A(x_0)Z_t$ is also transient.

4. Existence.

In this section we prove there exists a weak solution to the SDE (1.1) and that any weak solution to the SDE (1.1) solves a martingale problem. Note that in [LM] an equivalence is proved between the existence of weak solutions to SDEs and the existence of solutions to the corresponding martingale problem, by using a martingale representation theorem (see II.1.c on p.74 of [LM]). In [Km1], Komatsu has obtained an existence result for martingale problems that is applicable to the operator $\mathcal{L}$ defined in (1.2). So combining [Km1] with [LM] is one method of proving the existence of a weak solution to the SDE (1.1).

For the reader's convenience and for the sake of completeness, we nevertheless give a *direct* proof of the existence of a weak solution to the SDE (1.1) here, without using results from [Km1] nor [LM]. Moreover, the proof of Theorem 4.3 will be used in the proof of the last part of Theorem 1.1; see Remark 6.4 below.

We begin with a calculation. In the first proposition, we do not need to assume $A(x)$ satisfies Assumption 2.1.

Proposition 4.1. *Suppose that $A(x)$ is bounded and measurable, $\mathcal{L}$ is defined by (2.3), and $\mathbb{P}$ is a weak solution to the SDE (1.1). If $f \in C_b^2$, then $f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds$ is a martingale.*

Proof. Let $\overline{Z}_t^j$ be Z_t^j with all jumps of size larger than 1 in absolute value removed. That is, $\overline{Z}_t^j = Z_t^j - \sum_{s \le t} \Delta Z_s^j 1_{\{|\Delta Z_s^j|>1\}}$. It is well known that a Lévy process with bounded jumps has finite moments of all orders; see [Be]. Using symmetry, we see that each $\overline{Z}_t^j$ is a martingale. By Ito's formula,

$$\begin{aligned} f(X_t) - f(X_0) & \qquad (4.1) \\ &= \int_0^t \nabla f(X_{s-}) \cdot dX_s + \sum_{s \le t} \Big(f(X_s) - f(X_{s-}) - \nabla f(X_{s-}) \cdot \Delta X_s \Big) \\ &= \int_0^t \nabla f(X_{s-}) A(X_{s-}) dZ_s + \sum_{s \le t} \Big(f(X_{s-} + A(X_{s-})\Delta Z_s) \\ &\qquad - f(X_{s-}) - \nabla f(X_{s-}) \cdot A(X_{s-})\Delta Z_s \Big) \end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \nabla f(X_{s-})A(X_{s-})d\overline{Z}_s + \sum_{s\le t}\Big(f(X_{s-}+A(X_{s-})\Delta Z_s) \\
&\qquad - f(X_{s-}) - \nabla f(X_{s-})\cdot A(X_{s-})\Delta Z_s 1_{\{|\Delta Z_s|\le 1\}}\Big) \\
&= \int_0^t \nabla f(X_{s-})A(X_{s-})d\overline{Z}_s + \sum_{j=1}^d\sum_{s\le t}\Big(f(X_{s-}+a_j(X_{s-})\Delta Z^j_s) \\
&\qquad - f(X_{s-}) - \nabla f(X_{s-})\cdot a_j(X_{s-})\Delta \overline{Z}^j_s\Big) \\
&:= \int_0^t \nabla f(X_{s-})A(X_{s-})d\overline{Z}_s + \sum_{j=1}^d B^j_t.
\end{aligned}
$$

Here $a_j(x)$ is the jth column of the matrix $A(x)$ and $a_j(X_{s-})\Delta\overline{Z}^j_s$ denotes the vector whose i^{th} component is $A_{ij}(X_{s-})\Delta\overline{Z}^j_s$. In the second to the last equality, we used the fact that since $Z^1,\cdots,Z^d$ are independent symmetric α-stable processes, with probability one at most one of the $Z^1,\cdots,Z^d$ can jump at any given time. Using the Lévy system formula for Z^j, we have that

$$
\begin{aligned}
B^j_t = \text{martingale } + \int_0^t\int_{\mathbb{R}\setminus\{0\}}\Big(f(X_{s-}+a_j(X_{s-})w) - f(X_{s-}) \\
- \nabla f(X_{s-})\cdot a_j(X_{s-})w1_{\{|w|\le 1\}}\Big)\frac{c_\alpha}{|w|^{1+\alpha}}\,dw.
\end{aligned} \tag{4.2}
$$

Therefore

$$
\sum_{j=1}^d B^j_t = \text{martingale } + \int_0^t \mathcal{L}f(X_{s-})ds = \text{martingale } + \int_0^t \mathcal{L}f(X_s)ds.
$$

The last equality follows because X_s has at most countably many jumps. This proves the proposition. □

Proposition 4.2. *Suppose that the $A_{ij}(x)$ is bounded Lipschitz continuous in $x\in\mathbb{R}^d$ for every $1\le i,j\le d$ and for each $x\in\mathbb{R}^d$ the matrix $A(x)=(A_{ij}(x))_{1\le i,j\le d}$ is nondegenerate. Then there exists a weak solution to the SDE (1.1).*

Proof. Fix M large for the moment and let $\overline{Z}^j_t$ be obtained from Z^j with jumps of size larger than M in absolute value removed. That is, $\overline{Z}^j$, $j=1,\cdots,d$, are independent Lévy processes on $\mathbb{R}$, each with Lévy measure $c_\alpha|w|^{-1-\alpha}1_{\{|w|\le M\}}$. As above, each $\overline{Z}^j_t$ is a martingale and has finite moments of all orders. In particular, there exists c_1 depending only on t, M, and α such that

$$
\mathbb{E}\left[|\overline{Z}^j_t|^4\right] \le c_1. \tag{4.3}
$$

A standard Picard iteration argument then proves the existence and uniqueness of a solution to

$$
dX^i_t(M) = \sum_{j=1}^d A_{ij}(X_{t-}(M))d\overline{Z}^j_t. \tag{4.4}
$$

Note that $X_t(M)$ also will be a solution to the SDE (1.1) for t less than the first time that one of the Z_t^j's has a jump larger than M in absolute value. If we define $X_t = X_t(M)$ for such times t, it is easy to check (using the uniqueness of the solution to (4.4)) that X_t will in fact be a solution to the SDE (1.1). It is also easy to check that if we let $\mathbb{P}'$ be the law of X_t, then $\mathbb{P}'$ is the required weak solution to the SDE (1.1). □

Theorem 4.3. *Suppose the $A_{ij}(x)$ satisfy Assumption 2.1; that is, $\{A_{ij}(x),\ 1 \le i,j \le d\}$ are bounded and continuous in $x \in \mathbb{R}^d$ and for each $x \in \mathbb{R}^d$ the matrix $A(x) = (A_{ij}(x))_{1\le i,j\le d}$ is nondegenerate. Then a weak solution to the SDE (1.1) exists.*

Proof. Let A_{ij}^n be a sequence of Lipschitz functions satisfying Assumption 2.1 with

$$\sup_{i,j,n} \|A_{ij}^n\|_\infty < \infty$$

and such that A_{ij}^n converges uniformly on compact sets to A_{ij}. Let $\mathbb{P}_n$ be a weak solution to the SDE (1.1) but with A_{ij} replaced by A_{ij}^n. Define $\mathcal{L}_n$ analogously to (2.3) but with A_{ij} replaced by A_{ij}^n.

Fix $n \ge 1$ for the moment and suppose $f \in C_b^2(\mathbb{R}^d)$. By Proposition 4.1, under $\mathbb{P}_n$ we have that $f(X_t) = f(X_0) +$ martingale $+ \int_0^t \mathcal{L}_n f(X_s)ds$. Note that

$$\begin{aligned}\mathcal{L}_n f(x) =& \sum_{j=1}^d \int_{\{|w|>1\}} \big(f(x + a_j^n(x)w) - f(x)\big) \frac{c_\alpha}{|w|^{1+\alpha}} dw \\ &+ \sum_{j=1}^d \int_{\{|w|\le 1\}} \big(f(x + a_j^n(x)w) - f(x) - w\nabla f(x)\cdot a_j^n(x)\big) \frac{c_\alpha}{|w|^{1+\alpha}} dw,\end{aligned}$$

where $a_j^n(x)$ is the j^{th} column of the matrix $A^n(x)$. Hence

$$|\mathcal{L}_n f(x)| \le c_1 \|f\|_\infty + c_1 \max_{1\le i,j\le d} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_\infty .$$

Therefore $\int_s^t |\mathcal{L}_n f(X_s)|ds$ is bounded by $c_f\,(t-s)$ for any $t > s \ge 0$, where c_f depends on f but not n.

We conclude $f(X_t) - f(X_0) - c_f t$ is a supermartingale under $\mathbb{P}_n$. We now apply the argument in [B2], Propositions 3.1 and 3.2, to conclude that the sequence $\mathbb{P}_n$ is tight in the space $\mathbb{D}$. Here $\mathbb{D}$ denotes the space of right continuous functions with left limits, equipped with Skorokhod topology; see [EK] for details. (The argument in [B2] is for one dimensional processes, but the argument goes through with no changes for higher dimensions.) We can find a subsequence n_j such that $\mathbb{P}_{n_j}$ converges weakly; call the limit $\mathbb{P}$.

Let

$$U_t^{i,+} := \sum_{s\le t} \Delta Z_s^i 1_{\{\Delta Z_t^i > 1\}}, \qquad U_t^{i,-} := -\sum_{s\le t} \Delta Z_s^i 1_{\{\Delta Z_t^i < -1\}},$$

and

$$\overline{Z}_t^i = Z_t^i - (U_t^{i,+} - U_t^{i,-}).$$

Note that each $\overline{Z}_t^i$ is a martingale that has finite moments of any order, $U_t^{i,+}$ and $U_t^{i,-}$ are increasing processes, and that these processes are all independent from each other. We write U_t^+ and U_t^- for the vectors whose i^{th} components are $U_t^{i,+}$ and $U_t^{i,-}$, respectively.

By Skorokhod's theorem (see [Bi]), we can find a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and processes $\{(X_t^n, \overline{Z}_t^n, V_t^{n,+}, V^{n,-}), n \geq 1\}$ and $(X_t', \overline{Z}_t', U_t^{+\prime}, U_t^{-\prime})$ such that the following hold: the law of $(X_t', \overline{Z}_t', U_t^{+\prime}, U_t^{-\prime})$ under $\mathbb{P}'$ is the same as the law of $(X_t, \overline{Z}_t, U_t^+, U_t^-)$ under $\mathbb{P}$; the law of (X_t^n, Z_t^n) under $\mathbb{P}'$ is the same as the law of $(X_t, (\overline{Z}_t + U_t^+ - U_t^-))$ under $\mathbb{P}_n$; and such that $(X_\cdot^{n_j}, \overline{Z}_\cdot^{n_j}, V_\cdot^{n_j,+}, V_\cdot^{n_j,-})$ converges a.s. to $(X_\cdot', \overline{Z}_\cdot', U_\cdot^{+\prime}, U_\cdot^{-\prime})$ in the space $\mathbb{D}$, where $V^{n,+}$ and $V^{n,-}$ are defined in exactly the same way as $U^{n,+}$ and $U^{n,-}$ are defined but with $\widetilde{Z}^n$ in place of Z. For notational simplification, we drop the primes and without loss of generality, we may take n_j to be n. Note that each component of $Z_t^n := \overline{Z}_t^n + V_t^{n,+} - V_t^{n,-}$ is an independent copy of a symmetric α-stable process on $\mathbb{R}$ and that (X^n, Z^n) solves the SDE (1.1) with $A^n(x)$ in place of $A(x)$. Clearly each component of $Z_t := \overline{Z}_t + U_t^+ - U_t^-$ is a symmetric α-stable process on $\mathbb{R}$ and the components are independent of each other. To show (X, Z) solves the SDE (1.1), we proceed as follows. For any ω such that $(X_\cdot^n(\omega), V_\cdot^{n,+}(\omega), V_\cdot^{n,-}(\omega))$ converges to $(X_\cdot(\omega), U_\cdot^+(\omega), U_\cdot^-(\omega))$ in the space $\mathbb{D}$, there is a sequence of increasing Lipschitz continuous functions $\{\lambda_n(t), n \geq 1\}$ depending on ω such that

$$\lim_{n\to\infty} \sup_{s\leq T} |\lambda_n(t) - t| = 0 \qquad \text{for every } T > 0.$$

and such that the uniform distance between $(X_t^n(\omega), V_t^{n,+}(\omega), V_t^{n,-}(\omega))$ and $(X_{\lambda_n(t)}(\omega), U_{\lambda_n(t)}^+(\omega), U_{\lambda_n(t)}^-(\omega))$ on each compact time intervals goes to zero as $n \to \infty$ (see Proposition 5.3 in Chapter 3 of [EK]). As $A^n(x)$ converges to $A(x)$ uniformly on each compact set, the above implies that for any $T > 0$,

$$\begin{aligned}\lim_{n\to\infty} \sup_{t\leq T} \Bigg| &\sum_{j=1}^d \int_0^t A_{ij}^n(X_{s-}^n(\omega)) d\left(V_s^{n,j,+}(\omega) - V_s^{n,j,-}(\omega)\right) \\ &- \sum_{j=1}^d \int_0^{\lambda_n(t)} A_{ij}^n(X_{s-}(\omega)) d\left(U_s^{j,+}(\omega) - U_s^{j,-}(\omega)\right) \Bigg| = 0.\end{aligned}$$

(Here $V_t^{n,j,+}$ is the j^{th} component of $V_t^{n,+}$ and similarly for $V_t^{n,j,-}$.) It follows that

$$\lim_{n\to\infty} \sum_{j=1}^d \int_0^t A_{ij}^n(X_{s-}^n) d(V_s^{n,j,+} - V_s^{n,j,-}) = \sum_{j=1}^d \int_0^t A_{ij}^n(X_{s-}) d(U_s^{j,+} - U_s^{j,-})$$

at any continuity point t for the right hand side. Observe that

$$\begin{aligned}&\Big|\sum_{j=1}^d \int_0^t A_{ij}^n(X_{s-}^n) d\overline{Z}_s^{n,j} - \sum_{j=1}^d \int_0^t A_{ij}(X_{s-}) d\overline{Z}_s^j\Big| \\ &\quad \leq \Big|\sum_{j=1}^d \int_0^t \left(A_{ij}^n(X_{s-}^n) - A_{ij}(X_{s-})\right) d\overline{Z}_s^{n,j}\Big| + \Big|\sum_{j=1}^d \int_0^t A_{ij}(X_{s-}) d(\overline{Z}_s^{n,j} - \overline{Z}_s^j)\Big|.\end{aligned}$$

Since $\overline{Z}^{n,j}$ and Z^j are Lévy processes obtained from a one-dimensional symmetric α-stable process with jumps of size larger that 1 in absolute value removed, they are square-integrable martingales. Recall that the predictable quadratic variation processes $\langle \overline{Z}^{n,j} \rangle$ and $\langle \overline{Z}^j \rangle$ of $\overline{Z}^{n,j}$ and $\overline{Z}^j$, respectively, are continuous increasing processes with zero initial value such that $(\overline{Z}^{n,j}_t)^2 - \langle \overline{Z}^{n,j} \rangle_t$ and $(\overline{Z}^j_t)^2 - \langle \overline{Z}^j \rangle_t$ are martingales, respectively. From the Lévy system formula, we see that for $t \geq 0$,

$$\langle \overline{Z}^{n,j} \rangle_t = \int_0^t \left(\int_{\{|h| \leq 1} \frac{c_\alpha h^2}{|h|^{1+\alpha}} dh \right) dt = \frac{2c_\alpha}{2-\alpha} t.$$

So

$$\begin{aligned}
\mathbb{E} \Big[\Big| \sum_{j=1}^d \int_0^t A^n_{ij}(X^n_{s-}) d\overline{Z}^{n,j}_s - \sum_{j=1}^d \int_0^t A_{ij}(X_{s-}) d\overline{Z}^j_s \Big|^2 \Big] \\
\leq & (2d) \sum_{j=1}^d \mathbb{E} \int_0^t \left(A^n_{ij}(X^n_{s-}) - A_{ij}(X_{s-}) \right)^2 d\langle \overline{Z}^{n,j} \rangle_s + (2d) \sum_{j=1}^d \mathbb{E} \int_0^t A_{ij}(X_{s-})^2 d\langle \overline{Z}^{n,j} - \overline{Z}^j \rangle_s \\
\leq & c_2 \sum_{j=1}^d \mathbb{E} \int_0^t \Big(A^n_{ij}(X^n_{s-}) - A_{ij}(X_{s-}) \Big)^2 ds + c_2 \sum_{j=1}^d \mathbb{E} \, (\overline{Z}^{n,j}_t - \overline{Z}^j_t)^2.
\end{aligned}$$

Using (4.3) and the fact that X^n_t and $\overline{Z}^n_t$ converge to X_t and $\overline{Z}_t$, respectively, in the space $\mathbb{D}$ almost surely, we have

$$\lim_{n \to \infty} \mathbb{E} \left(\overline{Z}^{n,j}_t - \overline{Z}^j_t \right)^2 = 0,$$

and for any $\varepsilon > 0$, there exists $l > 0$ such that

$$\mathbb{P} \left(\sup_{s \leq t} |X^n_s| \leq l \text{ for all } n \geq 1 \right) > 1 - \varepsilon.$$

Since $A^n(x)$ converges uniformly to $A(x)$ on $[-l, l]$, there is a constant $c_3 > 0$ that is independent of n such that

$$\lim_{n \to \infty} \sum_{j=1}^d \mathbb{E} \int_0^t \left(A^n_{ij}(X^n_{s-}) - A_{ij}(X_{s-}) \right)^2 ds \leq c_3 \, \varepsilon,$$

which implies that the limit is zero since ε can be arbitrarily small. Thus we conclude

$$\lim_{n \to \infty} \sum_{j=1}^d \int_0^t A^n_{ij}(X^n_{s-}) dZ^{n,j}_s = \sum_{j=1}^d \int_0^t A_{ij}(X_{s-}) dZ^j_s$$

almost surely for each fixed $t \geq 0$ and thus

$$X_t = x_0 + \int_0^t A(X_{s-}) dZ_s \quad \text{a.s.}$$

for every fixed $t \geq 0$. Since the processes on the left and right hand side of the last display are right continuous with left limits, we see that X_t solves the SDE (1.1) with $X_0 = 0$. □

As a corollary we have the following explicit tightness estimate.

Corollary 4.4. *If $\mathbb{P}$ is a weak solution to the SDE (1.1), then*

$$\mathbb{P}\left(\sup_{s\le t}|X_s - X_0| > \delta\right) \le c\,t/\delta^2 \qquad \textit{for every } \delta > 0.$$

Here $c > 0$ is a constant that depends only on the upper bounds for functions $|A_{ij}(x)|$, $1 \le i, j \le d$ and on the dimension d.

Proof. Exactly as in the first part of the proof of Theorem 4.3, if $f \in C_b^2$, then $f(X_t) - c_f t$ is a supermartingale for a constant c_f depending only on f. We now appeal to Proposition 3.1 of [B2]. □

Remark 4.5. It follows from Corollary 4.4 that

$$\lim_{\lambda\to\infty} \mathbb{P}\left(\sup_{s\le t}|X_s - X_0| > \lambda\right) = 0.$$

We now make a preliminary step concerning the question of weak uniqueness of the solution to the SDE (1.1). We convert the problem to one of uniqueness of a martingale problem.

Theorem 4.6. *Suppose $\mathbb{P}$ is a weak solution to the SDE (1.1). Then $\mathbb{P}$ is a solution to the martingale problem for the operator $\mathcal{L}$ given by (2.3).*

Proof. This is immediate from Proposition 4.1. □

If $\mathbb{P}_1$ and $\mathbb{P}_2$ are two weak solutions to the SDE (1.1), then this theorem says they are both solutions to the martingale problem for $\mathcal{L}$. If we prove uniqueness for the solution to the martingale problem, we then conclude weak uniqueness for the SDE (1.1). Also, by virtue of Theorem 4.6 we can conclude there exists a solution to the martingale problem for $\mathcal{L}$.

5. Boundedness of the resolvent.

Throughout this section we suppose the dimension $d \ge 2$. Suppose $\mathbb{P}$ is a solution to the martingale problem for $\mathcal{L}$ started at x_0. We will use $\mathbb{E}$ to denote expectation with respect to $\mathbb{P}$. Define

$$S_\lambda f := \mathbb{E}\int_0^\infty e^{-\lambda t} f(X_t)dt. \tag{5.1}$$

Our goal in this section is to prove that for $p > \overline{p}$, where $\overline{p}$ is the constant given by (3.24),

$$\sup_{\|f\|_p\le 1} |S_\lambda f| < \infty,$$

or equivalently,

$$|S_\lambda f| \le c_1 \,\|f\|_p,$$

provided

$$\eta = \sup_{i,j} \|A_{ij}(x) - A_{ij}(x_0)\|_\infty \tag{5.2}$$

is small.

Let R_λ be the resolvent operator for the Lévy process $U_t = U_0 + A(x_0)Z_t$. For each n define the process Y^n_t to be equal to $X_{k/2^n}$ if $k/2^n \le t < (k+1)/2^n$ and $t \le n$. If $t > n$, define $Y_t = X^n_n$. Next define U^n_t as the solution to

$$dU^n_t = A(Y^n_{t-})dZ_t, \qquad U^n_0 = x_0. \tag{5.3}$$

Since the Y^n_t are piecewise constant and constant after time n, then U^n_t is uniquely defined. Define

$$V^n_\lambda f := \mathbb{E}\int_0^\infty e^{-\lambda t} f(U^n_t)dt. \tag{5.4}$$

Lemma 5.1. *(a) If f is continuous and bounded, then $V^n_\lambda f \to S_\lambda f$.*

(b) If $p > \max\{1, d/\alpha\}$, then for each $n \ge 1$, $\sup_{\|f\|_p \le 1} |V^n_\lambda f| < \infty$. That is, $|V^n_\lambda f| \le c\,\|f\|_p$ for some constant $c_1 = c_1(n, d, \alpha, p, \lambda) < \infty$.

Proof. (a) It follows from the definition that Y^n_{t-} converges to X_{t-} as $n \to \infty$. Hence U^n_t converges to $x_0 + \int_0^t A(X_{s-})dZ_s = X_t$. By dominated convergence, $V^n_\lambda f \to S_\lambda f$.

(b) Suppose $f \in L^p$. Over the time interval $[0, 1/2^n]$ the process $Y^n_t = x_0$, so U^n_t is equal in law to the process U_t started at x_0. Therefore

$$\Big|\mathbb{E}\int_0^{1/2^n} e^{-\lambda t} f(U^n_t)dt\Big| = \Big|\mathbb{E}\int_0^{1/2^n} e^{-\lambda t} f(U_t)dt\Big|.$$

The right hand side is

$$\Big|\mathbb{E}^{x_0}\int_0^{1/2^n} e^{-\lambda t} f(U_t)dt\Big| \le \mathbb{E}^{x_0}\int_0^\infty e^{-\lambda t}|f(U_t)|dt = R_\lambda(|f|)(x_0). \tag{5.5}$$

By Proposition 2.2(d), this is bounded by $c_2\|f\|_p$.

We now look at

$$\mathbb{E}\int_{k/2^n}^{(k+1)/2^n} e^{-\lambda t} f(U^n_t)dt.$$

Over the time interval $[k/2^n, (k+1)/2^n]$ the process $U^n_t = U^n_{k/2^n} + A(X_{k/2^n})Z_{t-k/2^n}$. If R^n_λ is the resolvent for U^n_t, we have

$$\begin{aligned}\Big|\mathbb{E}\int_{k/2^n}^{(k+1)/2^n} e^{-\lambda t} f(U^n_t)dt\Big| &= e^{-\lambda k/2^n}\Big|\mathbb{E}\Big[\mathbb{E}^{U^n_{k/2^n}}\int_0^{1/2^n} e^{-\lambda t} f(U^n_t)dt\Big]\Big| \\ &\le e^{-\lambda k/2^n}\sup_z \mathbb{E}^z\int_0^{1/2^n} e^{-\lambda t}|f(U^n_t)|dt \\ &\le e^{-\lambda k/2^n}\sup_z R^n_\lambda(|f|)(z).\end{aligned}$$

Using Proposition 2.2(d), this is bounded by $c_2\, e^{-\lambda k/2^n}\,\|f\|_p$.

Finally, we use the triangle inequality to obtain

$$|V_\lambda^n f| \le \sum_k \Big|\mathbb{E}\int_{k/2^n}^{(k+1)/2^n} e^{-\lambda t} f(U_t^n)dt\Big| \le c_2\|f\|_p \sum_k e^{-\lambda k/2^n} = \frac{c_2}{1-e^{-\lambda/2^n}}\,\|f\|_p.$$

This proves the lemma with $c_1 = c_2/(1-e^{-\lambda/2^n})$. □

Much harder is to give a uniform bound for $\sup_{\|f\|_p\le 1}|V_\lambda^n f|$. Define

$$\mathcal{B}f(x) := \mathcal{L}f(x) - \mathcal{L}_0 f(x).$$

Let $D_{\cdot j}(s,\omega)$ be the vector whose i^{th} component is $A_{ij}(X_{k/2^n-}(\omega))$ if $k/2^n \le s < (k+1)/2^n$ and $s \le n$ and $A(X_{n-}(\omega))$ if $s \ge n$. Define $\widetilde{\mathcal{L}}$ by

$$\widetilde{\mathcal{L}}f(x,s,\omega)$$
$$:= \sum_{j=1}^d \int_{\mathbb{R}\setminus\{0\}} \left[f(x+D_{\cdot j}(s,\omega)u) - f(x) - u1_{\{|u|\le 1\}}\nabla f(x)\cdot D_{\cdot j}(s,\omega)\right]\frac{c_\alpha}{|u|^{1+\alpha}}du,$$

and

$$\widetilde{\mathcal{B}}f(x,s,\omega) := \widetilde{\mathcal{L}}f(x,s,\omega) - \mathcal{L}_0 f(x).$$

Note that both $\widetilde{\mathcal{L}}f(x,s,\omega)$ and $\widetilde{\mathcal{B}}f(x,s,\omega)$ are random.

Recall that $a_j(x)$ denotes the j^{th} column of the matrix $A(x) = (A_{ij}(x))_{1\le i,j\le d}$. Recall also the definitions of $\mathcal{G}_{a_0,\delta}$ from (3.31), the constant $\overline{p} > 1$ from Theorem 3.10, and the definition of η from (5.2); η is the quantity that measures the oscillation of the matrix-valued function $A(x)$.

Proposition 5.2. *Let $p > \overline{p}$ and $d \ge 2$. Set*

$$\mathcal{N}_\delta f(x) = \sum_{j=1}^d |\mathcal{G}_{a_j(x_0),\delta} f(x)|.$$

(a) *There exists $\delta_0 > 0$ such that if $\delta < \delta_0$, then*

$$\|\mathcal{N}_\delta f\|_p \le \tfrac{1}{4}\|f\|_p.$$

(b) *There exists $\eta_0 > 0$ such that if $\eta \le \eta_0$, then*

$$|\mathcal{B}R_0 f(x)| \le \mathcal{N}_\delta f(x) \quad \text{and} \quad |\widetilde{\mathcal{B}}R_0 f(x,s,\omega)| \le \mathcal{N}_\delta f(x)$$

for all $x \in \mathbb{R}^d, s \in [0,\infty)$, and $\omega \in \Omega$.

Proof. (a) is essentially just a restatement of Theorem 3.11. (b) follows from the definitions of $\mathcal{B}, \widetilde{\mathcal{B}}$, and $\mathcal{N}_\delta$. For example, for every $x \in \mathbb{R}^d$,

$$|\mathcal{B}R_0 f(x)| = |\mathcal{L}R_0 f(x) - \mathcal{L}_0 R_0 f(x)| \le \sum_{j=1}^d \left|\mathcal{K}_{a_j(x)} f(x) - \mathcal{K}_{a_j(x_0)} f(x)\right| \le \mathcal{N}_\delta f(x).$$

The inequality for $|\widetilde{\mathcal{B}}R_0 f(x,s,\omega)|$ can be proved similarly. $\square$

Theorem 5.3. *Suppose $p > \overline{p}$ and $d \ge 2$.*

(a) There exist constants c_1 and η_0 independent of n such that if $\eta \le \eta_0$, then

$$|V_\lambda^n f| \le c_1 \, \|f\|_p.$$

(b) We have

$$|S_\lambda f| \le c_1 \, \|f\|_p.$$

Proof. (b) follows from (a), Lemma 5.1(a), and Fatou's lemma. So it suffices to prove (a).

Let $\overline{Z}_t^j$ be Z_t^j with all jumps larger than 1 in absolute value removed. Suppose $f \in C_b^2$ and apply Ito's lemma to U_t^n. Let $D_{\cdot j}(s) = D_{\cdot j}(\omega, s)$ be the vector whose i^{th} component is $A_{ij}(X_{k/2^n-}(\omega))$ if $k/2^n \le s < (k+1)/2^n$ and $s \le n$ and $A(X_{n-}(\omega))$ if $s \ge n$. Note that since $Z^1, \cdots, Z^d$ are independent symmetric α-stable processes, with probability one at most one of the $Z^1, \cdots, Z^d$ can jump at any given time. We have by Ito's formula,

$$\begin{aligned}
&f(U_t^n) - f(U_0^n)\\
&= \int_0^t \nabla f(U_{s-}^n) dU_s^n + \sum_{s \le t} \Big(f(U_s^n) - f(U_{s-}^n) - \nabla f(U_{s-}^n) \cdot \Delta U_s^n \Big)\\
&= \sum_{j=1}^d \int_0^t \nabla f(U_{s-}^n) D_{\cdot j}(s) dZ_s^j\\
&\qquad + \sum_{j=1}^d \sum_{s \le t} \Big(f(U_{s-}^n + D_{\cdot j}(s)\Delta Z_s^j) - f(U_{s-}^n) - \nabla f(U_{s-}^n) \cdot D_{\cdot j}(s) \Delta Z_s^j \Big)\\
&= \sum_{j=1}^d \int_0^t \nabla f(U_{s-}^n) D_{\cdot j}(s) d\overline{Z}_s^j\\
&\qquad + \sum_{j=1}^d \sum_{s \le t} \Big(f(U_{s-}^n + D_{\cdot j}(s)\Delta Z_s^j) - f(U_{s-}^n) - \nabla f(U_{s-}^n) \cdot D_{\cdot j}(s) \Delta Z_s^j 1_{\{|\Delta Z_s^j| \le 1\}} \Big)\\
&= \text{martingale}\\
&\quad + \sum_{j=1}^d \int_0^t \int_{\mathbb{R}\setminus\{0\}} \Big(f(U_{s-}^n + D_{\cdot j}(s)u) - f(U_{s-}^n) - u 1_{\{|u|\le 1\}} \nabla f(U_{s-}^n) \cdot D_{\cdot j}(s) \Big) \frac{c_\alpha}{|u|^{1+\alpha}} du.
\end{aligned}$$

If we take expectations with respect to $\mathbb{P}$, we have

$$\mathbb{E}\, f(U_t^n) = \mathbb{E}\, f(U_0^n) + \mathbb{E} \int_0^t \widetilde{\mathcal{L}} f(U_{s-}^n(\omega), s, \omega) ds.$$

Multiply by $e^{-\lambda t}$ and integrate over t from 0 to ∞. We obtain

$$\begin{aligned}
V_\lambda^n f &= \frac{1}{\lambda} \mathbb{E}\, f(U_0^n) + \mathbb{E} \int_0^\infty e^{-\lambda t} \int_0^t \widetilde{\mathcal{L}} f(U_s^n(\omega), s, \omega) ds\, dt\\
&= \frac{1}{\lambda} \mathbb{E}\, f(U_0^n) + \mathbb{E} \int_0^\infty \widetilde{\mathcal{L}} f(U_s^n(\omega), s, \omega) ds \int_s^\infty e^{-\lambda t} dt\, ds\\
&= \frac{1}{\lambda} \mathbb{E}\, f(U_0^n) + \frac{1}{\lambda} \mathbb{E} \int_0^\infty e^{-\lambda s} \widetilde{\mathcal{L}} f(U_s^n(\omega), s, \omega) ds.
\end{aligned}$$

If $g \in C_b^2$ with compact support, then $f = R_\lambda g$ will also be in C_b^2 by translation invariance. We have

$$\widetilde{\mathcal{L}}f(x,s,\omega) = \widetilde{\mathcal{B}}R_\lambda g(x,s,\omega) + \mathcal{L}_0 f(x) = \widetilde{\mathcal{B}}R_\lambda g(x,s,\omega) + \lambda R_\lambda g(x) - g(x).$$

We thus have

$$V_\lambda^n R_\lambda g = \frac{1}{\lambda}\mathbb{E}\,R_\lambda g(U_0^n) + \frac{1}{\lambda}\mathbb{E}\int_0^\infty e^{-\lambda s}\widetilde{\mathcal{B}}R_\lambda g(U_s^n(\omega),s,\omega)ds + V_\lambda^n R_\lambda g - \frac{1}{\lambda}V_\lambda^n g.$$

Rearranging,

$$V_\lambda^n g = R_\lambda g(x_0) + \mathbb{E}\int_0^\infty e^{-\lambda s}\widetilde{\mathcal{B}}R_\lambda g(U_s^n(\omega),s,\omega)ds. \tag{5.6}$$

Let $h = g - \lambda R_\lambda g$. By Remark 3.12, $A(x_0)Z_t$ is transient. Since g is bounded with compact support, $R_0 g$ is well defined. Observe that

$$\int_0^N P_t(R_\lambda g)dt = R_\lambda\Big(\int_0^N P_t g dt\Big).$$

The integral inside the parentheses on the right hand side converges boundedly and pointwise to $R_0 g$ as $N \to \infty$, therefore the expression on the left also converges, and so $R_0(R_\lambda g)$ is well defined. A simple calculation shows that $R_0 h = R_\lambda g$. Note that

$$\|h\|_p \le \|g\|_p + \|\lambda R_\lambda g\|_p \le 2\|g\|_p \tag{5.7}$$

by Proposition 2.2(b). So to show $|V_\lambda^n g|$ is bounded by a constant times $\|g\|_p$, it suffices to show that

$$\mathbb{E}\int_0^\infty e^{-\lambda s}\widetilde{\mathcal{B}}R_0 h(U_s^n(\omega),s,\omega)ds$$

is bounded by a constant times $\|h\|_p$.

Choose δ so that the conclusion of Proposition 5.2(a) holds and then η_0 so that the conclusion of Proposition 5.2(b) holds. We then have

$$\begin{aligned}\Big|\mathbb{E}\int_0^\infty e^{-\lambda s}\widetilde{\mathcal{B}}R_0 h(U_s^n,s)ds\Big| &\le \mathbb{E}\int_0^\infty e^{-\lambda s}\mathcal{N}_\delta h(U_s^n)ds\\ &= V_\lambda^n(\mathcal{N}_\delta h).\end{aligned}$$

We have shown in Lemma 5.1 that $|V_\lambda^n g| \le c_2\,\|g\|_p$ where c_2 depends on n but not g. Let

$$\Theta_n = \sup_{\|g\|_p \le 1} |V_\lambda^n g|.$$

Therefore $\Theta_n \le c_2 < \infty$. By (5.6),(5.7), Proposition 2.2(c), and Proposition 5.2(a)

$$\begin{aligned}|V_\lambda^n g| &\le c_3\|g\|_p + (V_\lambda^n(\mathcal{N}_\delta h))\\ &\le c_3\|g\|_p + \Theta_n\,(\|\mathcal{N}_\delta h\|_p)\\ &\le \big(c_3 + \tfrac{1}{2}\Theta_n\big)\,\|g\|_p.\end{aligned}$$

Taking the supremum over g such that $\|g\|_p \le 1$ and $g \in C_b^2$ with compact support,

$$\Theta_n \le c_3 + \tfrac{1}{2}\Theta_n.$$

Since $\Theta_n < \infty$, we conclude that

$$\Theta_n \le 2c_3,$$

a bound independent of n. □

6. Uniqueness.

In this section we show that for any point $x_0 \in \mathbb{R}^d$, the martingale problem for $\mathcal{L}$ started at x_0 has a unique solution. We continue to assume Assumption 2.1. Throughout this section the dimension $d \ge 2$.

Let $\mathbb{P}$ a solution to the martingale problem for $\mathcal{L}$ started at x_0 and let S_λ and R_λ be as in Section 5.

Proposition 6.1. *For $f \in C_b^2$,*

$$S_\lambda f = R_\lambda f(x_0) + S_\lambda \mathcal{B} R_\lambda f,$$

where $\mathcal{B} = \mathcal{L} - \mathcal{L}_0$.

Proof. Since $\mathbb{P}$ is a solution to the martingale problem, if $g \in C_b^2$, then $g(X_t) - g(X_0) - \int_0^t \mathcal{L}g(X_s)ds$ is a martingale, or

$$\mathbb{E}\, g(X_t) = \mathbb{E}\, g(X_0) + \mathbb{E} \int_0^t \mathcal{L}g(X_s)ds.$$

Multiply both sides by $e^{-\lambda t}$ and integrate over t from 0 to ∞. We then have

$$S_\lambda g = \frac{1}{\lambda} g(x_0) + \mathbb{E} \int_0^\infty \int_0^t e^{-\lambda t} \mathcal{L}g(X_s)ds\, dt.$$

Interchanging the order of integration, this is

$$\begin{aligned} S_\lambda g &= \frac{1}{\lambda} g(x_0) + \mathbb{E} \int_0^\infty \mathcal{L}g(X_s) \int_s^\infty e^{-\lambda t} dt\, ds \\ &= \frac{1}{\lambda} g(x_0) + \frac{1}{\lambda} \mathbb{E} \int_0^\infty e^{-\lambda s} \mathcal{L}g(X_s) ds \\ &= \frac{1}{\lambda} g(x_0) + \frac{1}{\lambda} S_\lambda \mathcal{L} g. \end{aligned}$$

Next set $g = R_\lambda f$ for $f \in C_b^2$. Note

$$\mathcal{L} R_\lambda f = \mathcal{L}_0 R_\lambda f + \mathcal{B} R_\lambda f = \lambda R_\lambda f - f + \mathcal{B} R_\lambda f.$$

Substituting and rearranging, we have

$$S_\lambda f = R_\lambda f(x_0) + S_\lambda \mathcal{B} R_\lambda f.$$

□

In the following statement η is the quantity defined in (5.2) that measures the oscillation of the matrix-valued function $A(x)$.

Proposition 6.2. *Let $x_0 \in \mathbb{R}^d$, where $d \geq 2$. There exists a constant η_0, depending only on $\mu_0(A, x_0)$ defined in (2.1) and on $\sup_{i,j} \sup_x |A_{ij}(x)|$, such that if $\eta < \eta_0$, then for any two solutions $\mathbb{P}_1$ and $\mathbb{P}_2$ to the martingale problem for $\mathcal{L}$ started at x_0, one has $\mathbb{P}_1 = \mathbb{P}_2$.*

Proof. Take $p > \overline{p}$. Let S^1_λ and S^2_λ be defined as in (5.1) with $\mathbb{P}_1$ and $\mathbb{P}_2$ in place of $\mathbb{P}$, respectively. Define $S^\Delta_\lambda g := S^1_\lambda g - S^2_\lambda g$. Let

$$\Theta = \sup_{\|g\|_p \leq 1} |S^\Delta_\lambda g|.$$

By Theorem 5.3(b), we have that Θ is finite. If we apply Proposition 6.1 with $\mathbb{P}_1$ and $\mathbb{P}_2$ and take the difference, we have for $f \in C^2_b$

$$S^\Delta_\lambda f = S^\Delta_\lambda \mathcal{B} R_\lambda f. \tag{6.1}$$

Take δ as in Proposition 5.2(a) and suppose η_0 is small enough so that the conclusion of Proposition 5.2(b) holds. By Proposition 5.2(b)

$$|\mathcal{B} R_0 h(x)| \leq \mathcal{N}_\delta h(x)$$

and so $\|\mathcal{B} R_0 h\|_p \leq \frac{1}{4}\|h\|_p$.

Let $h = f - \lambda R_\lambda f$, so that $R_\lambda f = R_0 h$. Then $\|h\|_p \leq 2\|f\|_p$. We now have

$$|S^\Delta_\lambda \mathcal{B} R_\lambda f| \leq \Theta \|\mathcal{B} R_\lambda f\|_p = \Theta \|\mathcal{B} R_0 h\|_p \leq \tfrac{1}{4}\Theta \|h\|_p \leq \tfrac{1}{2}\Theta \|f\|_p.$$

Substituting in (6.1) and taking the supremum over $f \in L^p$ with $\|f\|_p \leq 1$ and $f \in C^2_b$ with compact support, we see that

$$\Theta \leq \tfrac{1}{2}\Theta.$$

Since Θ is finite, we conclude $\Theta = 0$. Therefore for any $f \in C^2_b$ and $\lambda > 0$, $S^1_\lambda f = S^2_\lambda f$. By the uniqueness of Laplace transform, we have $\mathbb{E}_1 f(X_t) = \mathbb{E}_2 f(X_t)$ for almost every t. Since the paths of X_t are right continuous and f is continuous, then we have equality for all t. That $\mathbb{P}_1 = \mathbb{P}_2$ now follows by standard arguments; see, e.g., Chapter VI of [B3]. □

Theorem 6.3. *Suppose $d \geq 2$. Suppose for each $1 \leq i, j \leq d$ the function $A_{ij}(x)$ is bounded and continuous in x and for each $x \in \mathbb{R}^d$ the matrix $A(x)$ is nondegenerate. For every $x_0 \in \mathbb{R}^d$, there is a unique solution to the martingale problem for $\mathcal{L}$ started at x_0.*

Proof. Once we have Proposition 6.2 the theorem follows by standard arguments; see Proposition 5.2 and Section 6 of [B2]. See also Chapter VI of [B3], or Chapter 7 of [SV]. For the reader's convenience, we sketch the proof here.

Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be two solutions to the martingale problem for $\mathcal{L}$ started at x_0. Recall that $\mathbb{D}$ is the space of $\mathbb{R}^d$-valued right continuous functions on $[0, \infty)$ with left limits endowed with the Skorokhod topology. Define X_t to be the canonical process on $\mathbb{D}$ by setting $X_t(\omega) = \omega(t)$. For $N \geq 1$, let $\rho_N = \inf\{t : |X_t| > N\}$. Since both $\mathbb{P}_1$ and $\mathbb{P}_2$ are probability measures on $\mathbb{D}$, then under both $\mathbb{P}_1$ and $\mathbb{P}_2$ the process X_t has paths that are right continuous with left limits. It follows that $\rho_N \to \infty$ almost surely with respect to $\mathbb{P}_i$, $i = 1, 2$, as $N \to \infty$. Denote by $\mathcal{F}_t$ the σ-field generated by X_s for $s \le t$. To show that $\mathbb{P}_1 = \mathbb{P}_2$, it suffices to show that $\mathbb{P}_1 |_{\mathcal{F}_{\rho_N}} = \mathbb{P}_2 |_{\mathcal{F}_{\rho_N}}$ for each N large. Fix $N > |x_0| + 1$ and define

$$\|A\|_\infty := \max_{1\le i,j\le d} \sup_{x\in\mathbb{R}^d} |A_{ij}(x)|.$$

Since $A(x)$ is continuous and nondegenerate at each point, then

$$\mu_1(A, N) := \inf_{x\in B(0,N)} \inf_{u\in\mathbb{R}^d:\, |u|=1} |A(x)u| > 0.$$

Let η_0 be as in Proposition 6.2 but corresponding to $\frac{1}{2}\mu_1(A, N+1)$ and $2\|A\|_\infty$ (instead of $\mu_0(A, x_0)$ and $\|A\|_\infty$). Since $A(x)$ is uniformly continuous on the ball $B(0, N + 2)$, there exists $r \in (0, 1)$ such that

$$\sup_{1\le i,j\le d} |A_{ij}(x) - A_{ij}(y)| < \eta_0/2 \qquad \text{for } x, y \in B(0, N + 1) \text{ with } |x - y| < r.$$

One can find a continuous matrix-valued function $\widetilde{A}(x)$ such that $\widetilde{A}$ satisfies Assumption 2.1, $\widetilde{A}(x) = A(x)$ for $x \in B(0, r)$, is uniformly nondegenerate on $\mathbb{R}^d$ such that

$$\mu_2(\widetilde{A}) := \inf_{u\in\mathbb{R}^d:|u|=1} \inf_{x\in\mathbb{R}^d} |\widetilde{A}(x)u| > \mu_1(A, N)/2,$$

and

$$\sup_{x\in B(0,r)} \sup_{1\le i,j\le d} |\widetilde{A}_{ij}(x) - \widetilde{A}_{ij}(x_0)| < \eta_0.$$

By Theorem 4.3, Propositions 4.1 and 6.2, there is a unique solution to the martingale problem corresponding to $\widetilde{A}(x)$ starting at x_0 for every $x_0 \in \mathbb{R}^d$; call the solution $\widetilde{\mathbb{P}}^{x_0}$.

Recall that X_t is the canonical projection map on the space $\mathbb{D}$ of right continuous functions $\mathbb{R}^d$ having left limits. Let θ_t be the usual shift operators on $\mathbb{D}$ and let $\tau = \inf\{t > 0 : |X_t - x_0| \ge r\}$. Define

$$\mathbb{Q}_i(A \cap B \circ \theta_\tau) := \mathbb{E}_i \left[\widetilde{\mathbb{P}}^{X_\tau}(B);\, A\right], \qquad A \in \mathcal{F}_\tau,\ B \in \mathcal{F}_\infty.$$

It is straightforward to check that each $\mathbb{Q}_i$, $i = 1, 2$ solves the martingale problem corresponding to $\widetilde{A}$ started from x_0. So by Proposition 6.2, $\mathbb{Q}_1 = \mathbb{Q}_2$. This implies that $\mathbb{P}_1 = \mathbb{P}_2$ on $\mathcal{F}_\tau$.

Now define $\tau_1 := \tau$ and $\tau_{k+1} := \inf\{t > \tau_k : |X_t - X_{\tau_k}| > r\} \wedge \rho_N$ for $k \ge 1$. Corollary 4.4 implies that $\lim_{k\to\infty} \tau_k = \rho_N$. Using induction and regular conditional probabilities, it is easy to check that $\mathbb{P}_1 = \mathbb{P}_2$ on $\mathcal{F}_{\rho_N}$. □

Remark 6.4. For each $x \in \mathbb{R}^d$, the martingale problem for $\mathcal{L}$ started at x has a unique solution $\mathbb{P}^x$. It follows from this uniqueness and a standard regular conditional probability argument (see, e.g., Section 6 of [B2]) that $(X_t, \mathbb{P}^x, x \in \mathbb{R}^d)$ is a conservative strong Markov process on $\mathbb{R}^d$. Furthermore if $x_n \to x$, we see from the proof of Theorem 4.3 that $\{\mathbb{P}^{x_n}, n \ge 1\}$ is tight in the space $\mathbb{D}$ and any sequential limit solves the martingale problem for $\mathcal{L}$ started from x. Moreover, the proof shows that the marginal distribution of X_t under $\mathbb{P}^{x_n}$ converges to the marginal distribution of X_t under $\mathbb{P}^x$. This implies that $\lim_{n\to\infty} \mathbb{E}^{x_n} f(X_t) = \mathbb{E}^x f(X_t)$ for $f \in C_b$ and $t > 0$. In other words, the semigroup of $(X_t, \mathbb{P}^x, x \in \mathbb{R}^d)$ maps bounded continuous functions into bounded continuous functions.

Theorem 1.1 now follows by Theorem 6.3 and Remark 4.5 and Remark 6.4.

7. SDEs driven by spherically symmetric stable processes.

In this section we consider the system of stochastic differential equations

$$dX_t^i = \sum_{j=1}^d A_{ij}(X_{t-})dY_t^j, \qquad i = 1, \ldots, d, \quad \text{with } X_0 = x_0, \tag{7.1}$$

with $d \geq 2$ and $x_0 \in \mathbb{R}^d$. Here $Y = (Y^1, \cdots, Y^d)$ is a spherically symmetric α-stable process in $\mathbb{R}^d$ for some $0 < \alpha < 2$; that is, Y is a Lévy process in $\mathbb{R}^d$ with

$$\mathbb{E}\, e^{i\xi \cdot (Y_t - Y_0)} = e^{-t|\xi|^\alpha}, \qquad \xi \in \mathbb{R}^d.$$

We assume as before that $A_{ij}(x)$, $1 \leq i, j \leq d$, are continuous and bounded as functions of x and that for each $x \in \mathbb{R}^d$, the matrix $A(x) = (A_{ij}(x))$ is nondegenerate.

The purpose of this section is to sketch a proof of the following.

Theorem 7.1. *For each $x_0 \in \mathbb{R}^d$, there exists one and only one weak solution*

$$\{X = \{(X_t^1, \ldots, X_t^d),\, t \geq 0\},\ \mathbb{P}^{x_0}\}$$

to the SDE (7.1). The family $\{X,\ \mathbb{P}^x,\ x \in \mathbb{R}^d\}$ forms a conservative strong Markov process on $\mathbb{R}^d$ whose semigroup maps bounded continuous functions to bounded continuous functions.

As noted in the introduction, it can be shown similarly to Proposition 4.1 that the generator for any weak solution of the SDE (7.1) is

$$\widetilde{\mathcal{L}}f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left(f(x + A(x)y) - f(x) - \nabla f(x) \cdot (A(x)y) 1_{\{|y| \leq 1\}} \right) \frac{c_{d,\alpha}}{|y|^{d+\alpha}}\, dy,$$

and the symbol of $\widetilde{\mathcal{L}}$ is

$$\widetilde{\ell}(x, u) = -c_1 |A(x)u|^\alpha. \tag{7.2}$$

The proof for the existence of a weak solution is exactly analogous to that of Theorem 1.1 given in Section 4, so it is omitted here. The proof of the weak uniqueness assertion of Theorem 7.1 is much easier than the corresponding result for Theorem 1.1. We first note that it is well known that Proposition 2.2 holds for the process Y. Fix $x_0 \in \mathbb{R}^d$. Define $U_t = U_0 + A(x_0)Y_t$ and let $\mathbb{P}^x$ denote the law of U with $U_0 = x$; the expectation will be denoted by $\mathbb{E}^x$. For $\lambda \geq 0$, define

$$R_\lambda f(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} f(U_s) ds \qquad \text{for } f \in C_c(\mathbb{R}^d).$$

For a nondegenerate $d \times d$-matrix A, define $\widetilde{\mathcal{K}}_A$ as an operator on $C_c^2(\mathbb{R}^d)$ by

$$\widetilde{\mathcal{K}}_A f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left(R_0 f(x+Ay) - R_0 f(x) - \nabla R_0 f(x) \cdot (Ay) 1_{\{|y|\leq 1\}} \right) \frac{1}{|y|^{d+\alpha}} \, dy. \tag{7.3}$$

A direct calculation shows that

$$(\widehat{\widetilde{\mathcal{K}}_A f})(\xi) = -\frac{c_2 |A\xi|^\alpha}{|A(x_0)\xi|^\alpha} \, \widehat{f}(\xi). \tag{7.4}$$

Let $0 < \lambda_1 < \lambda_2$ be two positive constants. Suppose that $\lambda_1 I_{d\times d} \leq A(x_0) \leq \lambda_2 I_{d\times d}$; that is, for any $\xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d$,

$$\lambda_1 \sum_{i=1}^d \xi_i^2 \leq \sum_{i,j=1}^d A_{ij}(x_0) \xi_i \xi_j \leq \lambda_2 \sum_{i=1}^d \xi_i^2.$$

For any two constant matrices $A_1 := (a_{ij}^{(1)})$ and $A_2 = (a_{ij}^{(2)})$ such that

$$\lambda_1 I_{d\times d} \leq A_i \leq \lambda_2 I_{d\times d}, \qquad i = 1, 2,$$

the singular integral operator $\widetilde{\mathcal{K}}_{A_1} - \widetilde{\mathcal{K}}_{A_2}$ has symbol

$$h(\xi) := -c_2 \, \frac{|A_1\xi|^\alpha - |A_2\xi|^\alpha}{|A(x_0)\xi|^\alpha},$$

which is homogeneous in ξ of zero order and is C^∞ in $\{\xi : |\xi| > 1/2\}$. Note that the L^∞-norms of h and its derivatives up to second order with respect to ξ in the domain $\{\xi : |\xi| > 1\}$ are bounded by $c_3 \|A_1 - A_2\|$, where $c_3 > 0$ is a constant depending only on d, λ_1 and λ_2 and

$$\|A_1 - A_2\| := \sum_{i,j=1}^d |a_{ij}^{(1)} - a_{ij}^{(2)}|.$$

Thus by Theorem 3 of Calderón and Zygmund [CZ2], for any $p \geq 2$,

$$\|(\widetilde{\mathcal{K}}_{A_1} - \widetilde{\mathcal{K}}_{A_2}) f\|_p \leq c_3 c_4(p) \|A_1 - A_2\| \, \|f\|_p \qquad \text{for } f \in C_c^2(\mathbb{R}^d), \tag{7.5}$$

where $c_4(p) > 0$ is a constant that depends only on p. Estimate (7.5) is the exact analogue of Corollary 3.9. The weak uniqueness for solutions of (7.1) can then be established by following the arguments in Theorems 3.10 and 3.11 and in Sections 5 and 6. We omit the details here.

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