DIFFUSIONS ON THE SIERPINSKI CARPET

Richard F. BASS (Seattle)

ABSTRACT

This survey paper discusses the construction of a Brownian motion whose state space is the Sierpinski carpet. Harnack inequalities, connections with electrical resistance, and upper and lower bounds on the transition densities are also discussed. Some open problems are presented.

§1. Introduction

The Sierpinski carpet is the fractal formed by taking the unit square, dividing it into nine equal pieces, removing the central square, dividing each of the remaining eight little squares into nine pieces, and continuing. It is not hard to constrain Brownian motion or other random processes to be in the set F_n that is left when n stages of the construction have been done. But as more and more obstructions are introduced, the mean displacement gets smaller and smaller; in the limit one has only a process that moves not at all. If, however, one performs a renormalization at each stage, that is, if at the nth stage one speeds up the process deterministically and uniformly by an appropriate amount, one gets a nondegenerate limit, a random process that is called Brownian motion on the Sierpinski carpet.

The resulting process has some peculiar properties. For example, instead of the ubiquitous $t^{1/2}$ scaling for ordinary Brownian motion, one gets a type of scaling with exponent $d_s/2d_f$, where d_f is the fractal or Hausdorff dimension of the set, namely $\log 8/\log 3 \approx 1.89$, and d_s is a quantity (≈ 1.80) called the spectral dimension of the Sierpinski carpet, which is defined in terms of the asymptotics of the eigenvalues of the Laplacian on the Sierpinski carpet. Another peculiarity is that even though this process lives on a state space that can in no way be considered one dimensional, it still hits every point in the state space.

There are several links with other areas of probability and analysis. Not surprisingly, there are connections with random walks on graphs. Brownian motions on fractals are all symmetric processes and hence are associated with Dirichlet forms and Dirichlet spaces. If one considers the infinitesimal generator of these Brownian motions as the Laplacian, one can then study the heat equation on the carpet. Good heat kernel bounds have been found by using ideas from electrical circuit theory together with techniques associated with eigenvalue expansions.

There are also strong links with mathematical physics. There is a vast physics literature concerning random walks on fractals; see [39]. There are applications to the physics of disordered media, to field theories, and to the theory of diffraction, for example. Another reason physicists study diffusions on fractals is a conceptual one. To paraphrase Rammal and Toulouse [74], there are three dimensions that appear in physical theories: d_f, d_s, and the dimension of the Euclidean space in which the fractal is embedded. By coincidence, in usual physics all three dimensions are the same. Actually, the study of fractals has shown that the spectral dimension, d_s, is often the critical one.

We begin in Section 2 with a discussion of diffusions on the Sierpinski gasket and other finitely ramified fractals and how their study differs from that of diffusions on the Sierpinski carpet. Section 3 contains the notation we will need.

In Section 4 we begin the study of processes on the Sierpinski carpet. Let F_n be the nth stage of the construction of the carpet. We prove that with positive probability a Brownian motion $W_n(t)$ on F_n will exit F_n in certain places. This is used with a martingale argument in Section 5 to prove a uniform Harnack inequality for positive harmonic functions in F_n. If α_n is the largest the expected time to exit F_n is from any starting point and β_n is the smallest for points near the lower left hand corner of F_n, then there are some inequalities relating α_n, β_n, and α_{n-1}; these are proved in Section 6. In Section 7 we let $X_n(t) = W_n(\alpha_n t)$ and prove that $X_n(t)$ has subsequential limit points with respect to weak convergence and that any limit point X_t is a strong Markov process.

In Section 8 we turn to the connection of α_n with the solution of a certain eigenvalue problem and with the notion of electrical resistance. Section 9 uses these results to get some strong tightness estimates on $\mathbb{P}^x(\sup_{s\le t} |X_s - x| > \lambda)$.

We next turn to bounds on the transition densities of X_t. We obtain upper bounds in Section 10 and lower bounds in Section 11. Given these bounds we can say quite a bit about the path properties of X_t; see Section 12.

Section 13 discusses other fractals formed in a manner similar to that for the Sierpinski carpet and also how to construct Brownian motions on fractals embedded in $\mathbb{R}^d$, $d \ge 3$, where a coupling method is needed. Section 14 contains some further path properties and some generalizations of classical inequalities such as the Sobolev inequality. Finally, Section 15 mentions some open problems.

The letter c with subscripts denotes constants whose value is unimportant. We begin renumbering anew in each proposition, lemma, theorem, and corollary.

As this is a survey paper, many proofs are only sketched. We refer the reader to [5–10] for details.

§2. Finitely ramified fractals

The Sierpinski gasket is the fractal formed by first taking an equilateral triangle, dividing it into four equal subtriangles, removing the middle subtriangle, dividing each of the three remaining subtriangles into four, and continuing. Let G_n denote the set of points that are vertices of the 3^n triangles remaining at the nth stage. Goldstein [26], Kusuoka [56], and Barlow and Perkins [15] each constructed a process on the Sierpinski gasket by looking at a time change of the random walk on G_n and taking a weak limit. It is possible to calculate precisely what the time change should be, and Barlow and Perkins obtained good estimates on the transition densities of the limiting process.

Let X_n be the random walk on G_n. If $m > n$ and we look at X_m only at those times when it is at one of the points of G_n, it is not hard to see that the resulting process has the same in law as X_n. This property, called decimation invariance, is crucial to the proofs for the Sierpinski gasket.

There has been an explosion of results on Brownian motion on the Sierpinski gasket; see, for example, [23, 33, 34, 35, 36, 37, 38, 40, 42, 50, 53, 57, 61, 63, 68, 70, 72, 76, 77].

The Sierpinski gasket is an example of a finitely ramified fractal (cf. [80]). That is, it can be disconnected by removing finitely many points. There are many other finitely ramified fractals, and Brownian motions have been constructed for a large class of them. See the work of [2, 3, 4, 22, 24, 27, 41, 43, 44, 45, 46, 47, 48, 49, 52, 54, 55, 60, 64, 65, 66, 67, 75, 79]. Some of the calculations are not as explicit, but good estimates are still possible; see [14, 21, 30, 51, 52, 59, 68].

There has also been work on fractals that are formed in a random fashion and on the construction of Brownian motions on these random fractals. For example, see [12, 28, 29].

By contrast, the Sierpinski carpet is not finitely ramified, and there is no corresponding property to decimation invariance. Much less work has been done here. Osada [71] proved an isoperimetric inequality for approximations to the Sierpinski carpet. Kusuoka and Zhou [58] gave a construction of a self-similar Brownian motion on the Sierpinski carpet by means of Dirichlet forms theory and estimates of resistance and the constant in the Poincaré inequality. For more on the Sierpinski carpet, see also [13] and [25]. Other work will be discussed in what follows.

§3. Notation

Let $F_0 = [0,1]^2$ and let $F_1 = F_0 - (1/3, 2/3)^2$, that is, $[0,1]^2$ with the middle square removed. Let $\mathcal{S}_n$ denote the set of squares with side length 3^{-n} and vertices in $3^{-n}\mathbb{Z}^2$. Let Ψ_S be the affine map taking $[0,1]^2$ to the square S. We then let

$$F_2 = \bigcup\{\Psi_S(F_1) : S \subset F_1, S \in \mathcal{S}_1\}.$$

This is just a fancy way of saying that F_2 consists of what is left over if we take each of the eight squares making up F_1, divide each into nine equal subsquares, and remove the middle ones. We continue by letting $F_{n+1} = \bigcup\{\Psi_S(F_1) : S \subset F_n, S \in \mathcal{S}_n\}$, and then

$$F = \bigcap_{n=0}^{\infty} F_n. \tag{3.1}$$

The set F is the Sierpinski carpet ([81]). It is a closed set with zero Lebesgue measure and Hausdorff dimension $\log 8/\log 3$. We let μ_n be Lebesgue measure on F_n normalized to have total mass one and we let μ be the weak limit of the μ_n; then μ is a constant multiple of the Hausdorff-Besicovitch measure on F.

We also want an extension of F to $[0,\infty)$. We let

$$\widetilde{F} = \bigcup_{n=0}^{\infty} 3^n F. \tag{3.2}$$

We extend μ to $\widetilde{F}$ in the obvious way. The pre-carpet is the set defined by

$$\widetilde{F}_0 = \bigcup_{n=0}^{\infty} 3^n F_n. \tag{3.3}$$

We define $\partial_a F_n$ ("a" for absorbing) to be $F_n \cap \{(x,y) : x = 1 \text{ or } y = 1\}$. We let $\partial_r F_n$ ("r" for reflecting) be $\partial F_n - \partial_a F_n$, where we are using ∂A for the boundary of a set A. For a point $x \in F_0$, we let $D_n(x)$ be the square S of side length $2 \cdot 3^{-n}$ consisting of four squares of $\mathcal{S}_n$ such that x is closest to the center of S. Note that

$\partial D_{n+2}(x)$ is a positive distance from $\partial D_n(x)$. The open ball of radius r about x is denoted $B(x,r)$.

We let $W_n(t)$ be a Brownian motion on F_n with normal reflection on $\partial_r F_n$ and absorption on $\partial_a F_n$. Thus the process W_n is reflecting on the left and lower boundaries of F_n and is absorbed on the upper and right boundaries of F_n. For any process X, let

$$\tau = \tau(X) = \inf\{t : X_t \in \partial_a F_0\}.$$

Set

$$\sigma_r(x) = \sigma_r^X(x) = \inf\{t : X_t \notin D_r(x)\}.$$

If A is a Borel set, write

$$T_A = T_A(X) = \inf\{t : X_t \in A\}.$$

§4. Knight's moves

We begin by finding a lower bound on certain types of paths. First suppose we are looking at $W_n(t)$ on F_n with reflecting barriers on the x and y axes and absorption on the lines $\{x = 1\}$ and $\{y = 1\}$. Let

$$L_1 = [0, 1/2] \times \{1\}, \qquad L_2 = [1/2, 1] \times \{1\},$$

$$L_3 = \{1\} \times [1/2, 1], \qquad L_4 = \{1\} \times [0, 1/2].$$

We use symmetry and the reflection principle to obtain a lower bound on $\mathbb{P}^x(W_n(\tau) \in L_4)$ for certain points on the positive x axis.

Proposition 4.1. *If $x \in [0, 1/2] \times \{0\}$, then $\mathbb{P}^x(W_n(\tau) \in L_4) \geq 1/4$.*

Proof. F_n is symmetric about the line $\{y = x\}$. By the reflection principle, for any path started at x which exits F_n in $L_1 \cup L_2$, there is a corresponding path which exits F_n in $L_3 \cup L_4$. In addition there are paths from the starting point x to $L_3 \cup L_4$ that never hit the line $\{y = x\}$. Therefore

$$\mathbb{P}^x(W_n(\tau) \in L_3 \cup L_4) \geq \mathbb{P}^x(W_n(\tau) \in L_1 \cup L_2).$$

F_n is also symmetric about the line $\{y = 1/2\}$, and hence by the reflection principle again

$$\mathbb{P}^x(W_n(\tau) \in L_4) \geq \mathbb{P}^x(W_n(\tau) \in L_3).$$

Since $W_n(\tau)$ must be in $L_1 \cup L_2 \cup L_3 \cup L_4$, the proposition follows. □

Now consider the following configuration. Consider copies of F_n in the first, second, and third quadrants, formed by first reflecting F_n across the positive y axis and then reflecting the copy in the second quadrant across the negative x axis. We will make the positive x axis and the negative y axis reflecting and we will let τ be the first time $W_n(t)$ hits any of the lines $\{x = 1\}$, $\{x = -1\}$, $\{y = 1\}$, or $\{y = -1\}$. Let I_1 be the positive x axis, I_2 the positive y axis, I_3 the negative x axis, and I_4 the negative y axis. Let $T_0 = 0$ and let T_{i+1} be the first time after T_i that $W_n(t)$ hits one of the I_j different from the one $W_n(T_i)$ is in. Let Y_i be the I_j that $W_n(T_i)$ is in. So, for example, we might have $W_n(T_0) \in I_1, W_n(T_1) \in I_2, W_n(T_2) \in I_1, W_n(T_3) \in I_2, W_n(T_4) \in I_3$, etc., and then $Y_0 = 1, Y_1 = 2, Y_2 = 1, Y_3 = 2, Y_4 = 3$, and so on. It is clear that the Y_i form a Markov chain on the set $\{1, 2, 3, 4\}$, and in fact symmetry shows that Y_i is a simple symmetric random walk on $\{1, 2, 3, 4\}$ with reflection at 1 and 4.

The distribution of $W_n(\tau)$ is determined by two factors: (1) which quadrant $W_n(\tau)$ is in, and (2) given the quadrant, which piece of the boundary $W_n(\tau)$ lies in. From Markov chain considerations, there exists $c_1 > 0$, not depending on n, such that the probability that $W_n(\tau)$ lies in the first quadrant is greater than c_1 for all starting points along the positive x axis. By symmetry and Proposition 4.1, given that $W_n(\tau)$ lies in the first quadrant, there is probability at least 1/4 that $W_n(\tau) \in L_4$.

If we have one of the other configurations, namely, if we put copies of F_n in the first, second, third, and fourth quadrants, let τ be as before, and have either (1) no reflection on any axis, (2) reflection on the x axis, (3) reflection on the y axis, (4) reflection on both axes, (5) reflection on the negative y axis and negative x axis, or (6) reflection on the positive y axis and negative x axis, we get similar lower bounds.

We thus have the following proposition.

Proposition 4.2. *There exists c_1 not depending on n such that if $x \in [0, 1/2] \times \{0\}$, then*

$$\mathbb{P}^x(W_n(\tau) \in L_4) \geq c_1.$$

We call such a move from $[0, 1/2] \times \{0\}$ to L_4 a knight's move.

Similarly, we have lower bounds on corner moves. These are moves from $[1/2,1]\times\{0\}$ to L_4.

By combining a sequence of knight's and corner moves and using the strong Markov property and scaling, we see that given a curve in F_n, there is positive probability that $W_n(t)$ will stay close to the curve.

§5. Harnack inequality

From the knight's and corner moves we deduce a uniform Harnack inequality.

Theorem 5.1. *Suppose u is nonnegative and harmonic with respect to $W_n(t)$ in F_n. There exists c_1 independent of n such that*

$$u(y)\le c_1 u(x), \qquad x,y\in F_n\cap[0,1/2]^2.$$

Saying u is harmonic with respect to $W_n(t)$ means here that $u(W_n(t\wedge\tau))$ is a martingale. This is equivalent to saying that $\Delta u=0$ in F_n and $\partial u/\partial n=0$ a.e. on $\partial_r F_n$, where Δ is the Laplacian and $\partial u/\partial n$ is the normal derivative.

A Harnack inequality for F_n is well-known (see [17]). The crucial fact here is that c_1 does not depend on n.

Proof. Let $A\subseteq\partial_a F_n$ and $\lambda=\mathbb{P}^y(W_n(\tau)\in A)$. The process $M_t=\mathbb{P}^{W_n(t)}(W_n(\tau)\in A)$, $t\le\tau$, is a martingale bounded by 1. By some calculations using Doob's optional stopping theorem, with positive probability

$$\inf_{t\le\tau} M_t\ge\lambda/2.$$

If we take a path ω along which the infimum of M_t is greater than or equal to $\lambda/2$, then $\Gamma(t)=W_n(t)(\omega)$ is a curve in F_n connecting y to A such that $\mathbb{P}^z(W_n(\tau)\in A)\ge\lambda/2$ whenever $z\in\Gamma=\{\Gamma(t):0\le t\le\tau\}$.

Starting at x, by scaling, the strong Markov property, and knight's and corner move estimates, there exists c_2 independent of x and y and of n such that the curve $\{W_n(t):t\le\tau\}$ cuts off y from $\partial_a F_n$ with probability at least c_2. Therefore with $\mathbb{P}^x$ probability at least c_2, $W_n(t)$ hits Γ before hitting $\partial_a F_n$. By the strong Markov property,

$$\begin{aligned}\mathbb{P}^x(W_n(\tau)\in A)&\ge\mathbb{P}^x(W_n(\tau)\in A, T_\Gamma<\tau)\\&\ge\mathbb{E}^x\Big[\mathbb{P}^{W_n(T_\Gamma)}(W_n(\tau)\in A); T_\Gamma<\tau\Big]\\&\ge c_2\lambda/2.\end{aligned}$$

Hence

$$\mathbb{P}^y(W_n(\tau) \in A) \le (2/c_2)\mathbb{P}^x(W_n(\tau) \in A).$$

This proves Theorem 5.1 in the case $u(z) = \mathbb{E}^z 1_A(W_n(\tau))$. A limit argument yields the theorem for all nonnegative harmonic functions u. $\square$

Repeated applications of Theorem 5.1 and scaling imply that there exists c_1 not depending on n, r, or x such that if u is nonnegative and harmonic in $F_n \cap B(x,r)$, then

$$u(y) \ge c_1 u(x), \qquad y \in F_n \cap B(x, r/2). \tag{5.1}$$

An important consequence of the Harnack inequality is that it implies that harmonic functions are Hölder continuous. This argument is due to Moser [69].

Theorem 5.2. *Suppose u is harmonic and bounded in F_n. There exists c_1 and α not depending on u or n such that*

$$|u(x) - u(y)| \le c_1 |x-y|^\alpha \|u\|_\infty, \qquad x, y \in F_n \cap [0, 1/2]^2.$$

Proof. Define the oscillation of a function u on a set A by

$$\operatorname*{Osc}_A u = \sup_A u - \inf_A u.$$

It is easy to see that the theorem will be proved if we show there exists $\rho < 1$ such that for all r small enough so that $B(x,r) \cap [0,\infty)^2 \subseteq [0,1]^2$ for $x \in F_n \cap [0,1/2]^2$ we have

$$\operatorname*{Osc}_{B(x,r/2)\cap F_n} u \le \rho \operatorname*{Osc}_{B(x,r)\cap F_n} u.$$

By looking at $Au + B$ for suitable constants A and B we may suppose $\sup_{B(x,r)\cap F_n} u = 1$ and $\inf_{B(x,r)\cap F_n} u = 0$. By looking at $1 - u$ if necessary, we may suppose $u(x) \ge 1/2$. By (5.1), $u(y) \ge c_2 u(x) \ge c_2/2$ if $y \in B(x, r/2) \cap F_n$. Therefore the oscillation in $B(x, r/2) \cap F_n$ of u is less than $\rho = 1 - c_2/2$. $\square$

§6. Inequalities

We are going to time change $W_n(t)$ to obtain processes $X_n(t)$ and then in the next section we will show that the sequence X_n has a weak subsequential limit.

We let

$$\alpha_n = \sup_{x \in F_n} \mathbb{E}^x \tau(W_n).$$

If we then set $X_n(t) = W_n(\alpha_n t)$, we see that the expected time for X_n to exit F_n is less than or equal to 1. The difficulty is that there might be points y for which the time to exit is much less than 1, that is, the process moves almost instantaneously to $\partial_a F_n$. To show that this cannot happen, we let

$$\beta_n = \inf_{x \in F_n \cap [0,1/2]^2} \mathbb{E}^x \tau(W_n),$$

and we need to show that α_n / β_n remains bounded in n. This is where the Harnack inequality comes in.

Before showing this, let us derive some elementary relationships between the α_n and β_n.

Proposition 6.1. *There exists c_1 independent of n such that*

$$\beta_{n-1} \leq c_1 \beta_n.$$

Proof. To exit F_n, the process must first exit $F_n \cap [0, 1/3]^2$, which is equal to $(1/3)F_{n-1}$. Now use Brownian scaling. □

Only slightly harder is

Proposition 6.2. *There exists c_1 and r such that*

$$\alpha_n \leq c_1 \alpha_{n-1}, \qquad n \geq r.$$

Proof. By the knight's moves and corner moves, there exists r and η such that

$$\mathbb{P}^x(S_r > \tau) > \eta, \qquad x \in F_n \cap [0, 1/2]^2,$$

where S_r is the rth time W_n exits a square of the form $D_1(z)$; that is, $S_0 = 0$ and $S_{i+1} = \inf\{t > S_i : W_n(t) \notin D_1(W_n(S_i))\}$. For such x,

$$\begin{aligned} \mathbb{E}^x \tau &\leq \mathbb{E}^x[\tau; \tau > S_r] + \mathbb{E}^x[\tau; \tau \leq S_r] \\ &\leq \mathbb{E}^x\left[\mathbb{E}^{W_n(S_r)}\tau; \tau > S_r\right] + \mathbb{E}^x S_r \\ &\leq \alpha_n(1-\eta) + c_2 r \alpha_{n-1}. \end{aligned}$$

By use of the reflection principle as in Section 4, we see that $\mathbb{E}^x\tau$ takes its maximum in $F_n \cap [0,1/2]^2$. So taking the supremum over $x \in F_n \cap [0,1/2]^2$,

$$\alpha_n \leq \alpha_n(1-\eta) + c_3\alpha_{n-1}. \qquad \square$$

The key proposition is the following. Once we have this, we then have the existence of constants c_1, c_2, and c_3 such that

$$\alpha_n \leq c_1\alpha_{n-1} \leq c_2\beta_{n-1} \leq c_3\beta_n \leq c_3\alpha_n. \tag{6.1}$$

Proposition 6.3. *There exists c_1 such that $\alpha_n \leq c_1\beta_n$.*

The idea is to use induction on n. Let x and y be two points such that $\alpha_n = \mathbb{E}^y\tau$ and $\beta_n = \mathbb{E}^x\tau$. Starting at x, the time to exit F_n is controlled by two factors: the time to exit $D_1(x)$ and the time to go from $\partial D_1(x)$ to $\partial_a F_n$. The first factor is bounded in terms of α_{n-1}/β_{n-1}. Starting at a point in $\partial D_1(x)$, the knight and corner moves imply there is some chance $W_n(t)$ will first go near y, so the Harnack inequality can be used to bound the second factor. Here are the details.

Proof. Let $S_0 = 0$ and $S_{i+1} = \inf\{t > S_i : W_n(t) \notin D_1(W_n(S_i))\}$. By the knight and corner moves estimates, there exist r and η such that if $x, y \in F_n \cap [0,1/2]^2$, then

$$\mathbb{P}^x(W_n(S_r) \in D_3(y), S_r < \tau) > \eta.$$

Let $g(z) = \mathbb{E}^z\tau(W_n)$. The function $z \to \mathbb{E}^z g(W_n(\sigma_1(y)))$ is harmonic in $D_1(y)$, so by Theorem 5.1 there exists c_2 such that

$$\mathbb{E}^z g(W_n(\sigma_1(y))) \geq c_2\mathbb{E}^y g(W_n(\sigma_1(y))), \qquad z \in D_3(y).$$

We then have

$$\begin{aligned} \mathbb{E}^x g(W_n(\sigma_1(x))) &= \mathbb{E}^x(\tau - \sigma_1(x)) \\ &\geq \mathbb{E}^x(\tau - S_r; W_n(S_r) \in D_3(y), S_r < \tau) \\ &= \mathbb{E}^x\left[\mathbb{E}^{W_n(S_r)}\tau; W_n(S_r) \in D_3(y), S_r < \tau\right] \\ &\geq c_2\eta\mathbb{E}^y g(W_n(\sigma_1(y))). \end{aligned} \tag{6.2}$$

By the strong Markov property we can write

$$g(x) = \mathbb{E}^x\sigma_1(x) + \mathbb{E}^x g(W_n(\sigma_1(x))), \tag{6.3}$$

and similarly with x replaced by y. Using scaling,

$$\beta_{n-1}/9 \le \mathbb{E}^z \sigma_1(z) \le \alpha_{n-1}/9, \qquad z \in F_n \cap [0,1/2]^2. \tag{6.4}$$

If we let $r_n = \alpha_n/\beta_n$, we combine (6.2), (6.3), and (6.4) to obtain

$$\begin{aligned} g(y) &= \mathbb{E}^y \sigma_1(y) + \mathbb{E}^y g(W_n(\sigma_1(y))) \\ &\le \alpha_{n-1}/9 + (c_2\eta)^{-1}\mathbb{E}^x g(W_n(\sigma_1(x))) \\ &\le r_{n-1}(\beta_{n-1}/9) + (c_2\eta)^{-1}\mathbb{E}^x g(W_n(\sigma_1(x))) \\ &\le r_{n-1}\mathbb{E}^x \sigma_1(x) + (c_2\eta)^{-1}\mathbb{E}^x g(W_n(\sigma_1(x))). \end{aligned}$$

Choosing x and y so that $\alpha_n = g(y)$ and $\beta_n = g(x)$,

$$\alpha_n \le (r_{n-1} \vee (c_2\eta)^{-1})\beta_n,$$

or

$$r_n \le r_{n-1} \vee (c_2\eta)^{-1}. \tag{6.5}$$

The proposition follows by taking $c_1 = r_0 \vee (c_2\eta)^{-1}$. □

It is fortuitous that in (6.5) we have "$\vee$" rather than "+," for otherwise the proof would not work.

From (6.1) we can obtain a tightness estimate. The first step is to show

Proposition 6.4. *There exists $c_1 < 1$ such that*

$$\mathbb{P}^x(\tau \le s) \le c_1 + s/\alpha_n, \qquad x \in F_n \cap [0,1/2]^2.$$

Proof. We have

$$\begin{aligned} \mathbb{E}^x \tau &\le s + \mathbb{E}^x \left[\mathbb{E}^{W_n(s)}\tau; s < \tau\right] \\ &\le s + \alpha_n \mathbb{P}^x(s < \tau). \end{aligned}$$

On the other hand,

$$\mathbb{E}^x \tau \ge \beta_n \ge c_2 \alpha_n$$

with $c_2 < 1$. So $c_2\alpha_n \le s + \alpha_n \mathbb{P}^x(s < \tau)$ and the result follows with $c_1 = 1 - c_2$ by solving for $\mathbb{P}^x(\tau \le s) = 1 - \mathbb{P}^x(s < \tau)$. □

This proposition says that if γ is small enough, there is positive probability, say $\theta = \theta(\gamma)$, that W_n does not exit F_n before time $\gamma\alpha_n$. If we let $X_n(t) = W_n(\alpha_n t)$, then

$$\mathbb{P}^x(\tau(X_n) < \gamma) \le 1 - \theta.$$

This is enough to give tightness. Here is the idea; we give precise estimates in Section 9. Let $\varepsilon > 0$. Choose m such that $(1-\theta)^m < \varepsilon$. By scaling,

$$\mathbb{P}^x(\sigma_{2m}^{X_n}(x) < c_1(m)\gamma) \le 1 - \theta.$$

To exit F_n, X_n must cross at least m squares of the form $D_{2m}(y)$. So by the strong Markov property,

$$\mathbb{P}^x(\tau < c_1(m)\gamma) \le (1-\theta)^m \le \varepsilon.$$

Another scaling leads to

$$\mathbb{P}^x(\sigma_r^{X_n}(x) < c_2(r,m)\gamma) \le \varepsilon,$$

which is the tightness estimate.

§7. Constructing the process

Set

$$X_n(t) = W_n(\alpha_n t).$$

In other words we speed up W_n so that the largest the expected time to exit F_n can be is 1. Let

$$U_n^\lambda f(x) = \mathbb{E}^x \int_0^\tau e^{-\lambda t} f(X_n(t))\, dt, \tag{7.1}$$

and write $U_n f(x)$ for $U_n^0 f(x)$. We kill $X_n(t)$ when it hits $\partial_a F_n$.

Proposition 7.1. *If f is bounded, then $U_n f(x)$ is Hölder continuous with a bound that depends only on $\|f\|_\infty$ and $\|U_n f\|_\infty$ and not on n:*

$$|U_n f(x) - U_n f(y)| \le c_1 |x-y|^\alpha (\|f\|_\infty + \|U_n f\|_\infty).$$

Proof. Fix x_0 and for $x \in D_{r+2}(x_0)$ write

$$U_n f(x) = \mathbb{E}^x \int_0^{\sigma_r(x_0)} f(X_n(t))\, dt + \mathbb{E}^x U_n f(X_{\sigma_r(x_0)}).$$

By taking r large, we can make the first term on the right small. The second term on the right is harmonic in x for $x \in D_{r+2}(x_0)$ and so is Hölder continuous by Theorem 5.2. $\square$

Proposition 7.2. *If f is bounded, then $U_n^\lambda f(x)$ is Hölder continuous with a modulus that depends only on $\|f\|_\infty$ and λ.*

Proof. By the resolvent identity ([18]),

$$U_n^\lambda f = U_n(f - \lambda U_n^\lambda f).$$

Since $\|U_n^\lambda f\|_\infty \le \lambda^{-1}\|f\|_\infty$ and $\|f - \lambda U_n^\lambda f\|_\infty \le 2\|f\|_\infty$, the result follows from Proposition 7.1. $\square$

In order to have a constant state space, it is convenient to define $X_n(t)$ to be ordinary Brownian motion in $[0,1]^2 - F_n$ up until hitting F_n and then behaving according to the law of $W_n(\alpha_n t)$ thereafter.

Let $\{f_j\}$ be a countable dense subset of the bounded continuous functions on $[0,1]^2$ and let λ_i be a countable dense subset of $[0,\infty)$. By Proposition 7.2, for each i and j $\{U_n^{\lambda_i} f_j\}$ is an equicontinuous family in n. By a diagonalization procedure, there exists a subsequence n' such that $U_{n'}^{\lambda_i} f_j$ converges for each i and j. Since $\mathbb{E}^x\tau(X_{n'}) \le 1$, then $\|U_n^{\lambda_i}\|_\infty$ is uniformly bounded, and therefore $U_{n'}^{\lambda_i} f$ converges for each bounded continuous function f on $[0,1]^2$. Since $\|U_n^\lambda - U_n^\beta\|_\infty$ is small uniformly in n by the resolvent identity if $\lambda - \beta$ is small, it follows that $U_{n'}^\lambda f$ converges for each $\lambda \in [0,\infty)$. Let us call the limit $U^\lambda f$.

This resolvent convergence is enough to get a Markov process, but is not enough to get a continuous process. For that we use tightness. Let $\mathbb{P}_n^x$ denote the law of $X_n(t)$ started at x. Let X_t be the coordinate process on the set of continuous functions on $[0,1]^2$, that is, $X_t(\omega) = \omega(t)$.

Proposition 7.3. *If $x_{n'} \to x \in F$, then $\mathbb{P}_{n'}^{x_{n'}}$ converges weakly.*

Proof. By tightness, there is a subsequence of $\{n'\}$ that converges to a probability measure on the set of continuous functions on F. Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be two subsequential limit points. Since $\int_0^\tau e^{-\lambda t} f(X(t))\,dt$ is a continuous functional of the path of $X_n(t)$ when f is a continuous function on $[0,1]^2$, then for some subsequence $\{n_k\}$ of $\{n'\}$,

$$\mathbb{E}_{\mathbb{P}_1} \int_0^\tau e^{-\lambda t} f(X_t)\,dt = \lim_{k\to\infty} \mathbb{E}_{n_k}^{x_{n_k}} \int_0^\tau e^{-\lambda t} f(X_t)\,dt.$$

On the other hand, by the equicontinuity of $U_n^\lambda f$, this limit is also $U^\lambda f(x)$. This also is true for $\mathbb{P}_2$, so that the expectations of $\int_0^\tau e^{-\lambda t} f(X_t)\,dt$ under $\mathbb{P}_1$ and $\mathbb{P}_2$ are the same. By the uniqueness of the Laplace transform, the one-dimensional distributions of X_t under $\mathbb{P}_1$ and $\mathbb{P}_2$ are the same. A similar argument, using the fact that $(\mathbb{P}_n^x, X_n(t))$ are Markov processes, shows that the finite dimensional distributions of X_t under $\mathbb{P}_1$ and $\mathbb{P}_2$ are the same. Therefore $\mathbb{P}_1 = \mathbb{P}_2$, and this shows that $\mathbb{P}_{n'}^{x_{n'}}$ converges weakly. □

Let $\mathbb{P}^x$ be the weak limit of $\mathbb{P}_{n'}^x$. We know $x \to \mathbb{E}_n^x f(X_t)$ is continuous for each n if f is bounded and continuous. By the above proposition, $\mathbb{E}_{n'}^{x_{n'}} f(X_t)$ converges to $\mathbb{E}^x f(X_t)$ if $x_{n'}$ converges to x. A simple real variable argument then implies that $x \to \mathbb{E}^x f(X_t)$ is continuous and also that the convergence of $\mathbb{E}_{n'}^x f(X_t)$ to $\mathbb{E}^x f(X_t)$ is uniform. Together with the resolvent convergence, this allows us to conclude that $(\mathbb{P}^x, X_t)$ is a strong Markov process. By the tightness estimate, the paths of X_t are continuous under each $\mathbb{P}^x$. Since $\mathbb{E}^x \tau \le 1$, the process is not degenerate. Since each X_n is invariant under isometries of F_n, then so is X_t. We also have invariance under local isometries: up until exiting a subsquare of some size, the law of the process is invariant under rotations, reflections, and translations by suitable multiples of 3^{-m}. It is easy to see that if $x \in F$, then with $\mathbb{P}^x$ probability one the paths of X_t lie in F.

A standard piecing together argument allows us to extend X_t to all of $\widetilde{F}$, and we have a strong Markov process $(\mathbb{P}^x, X_t)$ with paths in $\widetilde{F}$.

One property we have not shown for our process X_t is self-similarity, namely, that $3X_t$ is equal in law to a deterministic time change of X_t. Kusuoka and Zhou [58] showed how one can achieve this. They construct processes $Y_n(t)$ on graphical approximations to F_n. Let $\mathcal{E}_n(f,f)$ be the Dirichlet form for $Y_n(t)$. Then Kusuoka and Zhou showed that $n^{-1}\sum_{m=1}^n \mathcal{E}_m(f,f)$ has subsequential limit points and that any subsequential limit point is a closable Dirichlet form on F. The process Y_t corresponding to any subsequential limit point then has the self-similarity property.

§8. Electrical resistance

One of the unsatisfactory aspects of the construction so far is that we have no good idea of what the constants α_n look like. In this section we show that the α_n are related to a certain eigenvalue problem, and that the eigenvalue problem is related in turn to an electrical resistance problem. As a result we will see that α_n

is comparable to $(8/9)^n \rho_F^n$ for a certain constant ρ_F.

Consider the eigenvalue problem for $-(1/2)\Delta$ on F_n with absorption on $\partial_a F_n$ and reflection on the remaining boundaries of ∂F_n. Let λ_n be the first eigenvalue for this situation and v_n the first eigenfunction, normalized so that $\sup_x v_n(x) = 1$. We show that α_n is comparable to λ_n^{-1}. Recall the definition of U_n from (7.1).

Proposition 8.1. *There exists c_1 such that*

$$c_1 \alpha_n \leq \lambda_n^{-1} \leq \alpha_n / 2.$$

Proof. To get the right hand inequality, we choose x_0 so that $v_n(x_0) = 1$ and write

$$2\lambda_n^{-1} = 2\lambda_n^{-1} v_n(x_0) = U_n v_n(x_0) \leq \|v_n\|_\infty \mathbb{E}^{x_0} \tau(X_n) \leq \alpha_n.$$

For the other inequality, let $h(x) = \mathbb{E}^x \tau(X_n)$. Since we have reflection on $\partial F_n - \partial_a F_n$, the normal derivative of h is zero almost everywhere there with respect to surface measure. By Green's first identity,

$$\int_{F_n} |\nabla h|^2 = 2 \int_{F_n} h,$$

since $h = U_n 1$. The first eigenvalue λ_n is given as the solution to a variational problem:

$$\lambda_n = \inf \Big\{ \int_{F_n} |\nabla f|^2 \Big/ \int_{F_n} f^2 \Big\}.$$

So

$$\lambda_n \leq \int_{F_n} |\nabla h|^2 \Big/ \int_{F_n} h^2.$$

Since

$$\int_{F_n} |\nabla h|^2 = 2 \int_{F_n} h \leq 2\alpha_n$$

and

$$\int_{F_n} h^2 \geq \int_{F_n \cap [0,1/2]^2} h^2 \geq \beta_n^2 / 4,$$

then $\lambda_n \leq 8\alpha_n / \beta_n^2$. We now use (6.1). □

We now define the resistance of F_n. We short out the left and right hand sides of F_n and put a unit potential across a conducting material in the shape of F_n and measure the voltage drop. Mathematically, we define the resistance ρ_n by

$$\rho_n^{-1} = \inf \Big\{ \int_{F_n} |\nabla u(x)|^2 \, dx : u = 0 \text{ on } \{x = 0\}, u = 1 \text{ on } \{x = 1\} \Big\}.$$

The quantity ρ_n^{-1} is called the conductance of F_n.

Proposition 8.2. *There exist c_1 and c_2 such that*

$$c_1\lambda_n \le (9/8)^n \rho_n^{-1} \le c_2\lambda_n.$$

Proof. Let h be as in Proposition 8.1 and set

$$g(x,y) = \frac{h(1-x,y)}{\beta_n} \wedge 1$$

for $y \le 1/2$ and $g(x,y) = g(x,1-y)$ for $y > 1/2$. Since $h(1,y) = 0$ and $h(0,y) \ge \beta_n$ if $y \le 1/2$, then $g(0,y) = 0, g(1,y) = 1$. So, recalling that the Lebesgue measure of F_n is $(8/9)^n$,

$$\begin{aligned}\rho_n^{-1} &\le \int_{F_n} |\nabla g|^2 = 2\int_{F_n\cap[0,1]\times[0,1/2]} |\nabla(1\wedge\beta_n^{-1}h)|^2 \\ &\le 2\beta_n^{-2}\int_{F_n}|\nabla h|^2 = 4\beta_n^{-2}\int_{F_n} h \\ &\le 4(8/9)^n\alpha_n/\beta_n^2.\end{aligned}$$

Now use (6.1) and Proposition 8.1 to get the right hand inequality.

The left hand inequality follows similar lines, although there is a complication in that the boundary conditions in the definition of ρ_n are on the left and right hand sides of F_n while the boundary conditions used in defining λ_n are on the right and top sides of F_n.

Let f_n be the function which yields the infimum in the definition of ρ_n^{-1}. For $y \le x$, let $e(x,y)$ be 1 if $x \le 1/3$, let $e(x,y) = 0$ if $x \ge 2/3$, and let $e(x,y) = f_{n-1}(2-3x,3y)$ if $1/3 \le x \le 2/3$. For $x < y$ define $e(x,y) = e(y,x)$. Then

$$\int_{F_n} |\nabla e|^2 \le 2\cdot\frac{9}{8}\int_{F_{n-1}} |\nabla f_{n-1}|^2 = \frac{9}{4}\rho_{n-1}^{-1},$$

while $\int_{F_n} e^2 \ge (1/8)(8/9)^n$. Therefore

$$\lambda_n \le \int_{F_n} |\nabla e|^2 \Big/ \int_{F_n} e^2 \le 18\rho_{n-1}^{-1}(9/8)^n,$$

and so

$$\lambda_{n-1} \le c_3\alpha_{n-1}^{-1} \le c_4\alpha_n^{-1} \le c_5\lambda_n \le 18c_5\rho_{n-1}^{-1}(9/8)^n. \qquad \square$$

The principal result of [7] is that there exists ρ_F and c_1 and c_2 such that

$$c_1\rho_F^n \le \rho_n \le c_2\rho_F^n. \tag{8.1}$$

We refer the reader to that paper for proofs. Here we say only that the proof uses subadditivity and notions from electrical circuit theory but no probability theory.

Proposition 8.3. $\rho_F \geq 7/6$.

Proof. Let a_i be the number of squares of side length $1/3$ in $F_1 \cap ([(i-1)/3, i/3] \times [0,1])$. So $a_1 = 3, a_2 = 2, a_3 = 3$ and $\sum_i a_i = 8$.

Now let us apply shorts along the lines $\{x = 1/3\}$ and $\{x = 2/3\}$. By scaling, the resistance across $F_n \cap [0, 1/3]^2$ is ρ_{n-1}. By Kirchoff's laws, the resistance between the lines $\{x = 0\}$ and $\{x = 1/3\}$ is $\rho_{n-1}/3$, and similarly, the resistance between the lines $\{x = 1/3\}$ and $\{x = 2/3\}$ is $\rho_{n-1}/2$ and the resistance between $\{x = 2/3\}$ and $\{x = 1\}$ is $\rho_{n-1}/3$. By Kirchoff's laws again

$$\rho_n \geq \rho_{n-1}\Big(\sum_{i=1}^{3} a_i^{-1}\Big) = (7/6)\rho_{n-1}.$$

By induction,

$$\rho_n \geq (7/6)^n \rho_0.$$

Using (8.1) gives the desired result. □

From Propositions 8.1, 8.2, and 8.3, we deduce $\alpha_n \to \infty$ as $n \to \infty$, which confirms our intuition.

Yuan and Tao [83, 84] conducted experiments to determine ρ_n, and obtained $\rho_F \approx 1.25$. Numerical calculations ([11]) show that the true value is ≈ 1.2515.

Suppose we had divided $[0,1] \times [0,1]$ into l_F^2 subsquares and had formed F_1 by removing all but m_F of them. In this case a similar argument would yield

$$\rho_F > l_F^2/m_F. \tag{8.2}$$

Define

$$d_s = \frac{2\log 8}{\log(8\rho_F)}, \qquad d_f = \frac{\log 8}{\log 3}.$$

For the more general case, the formulas are

$$d_s = \frac{2\log(m_F)}{\log(\rho_F m_F)}, \qquad d_f = \frac{\log m_F}{\log l_F}. \tag{8.3}$$

Define

$$d_w = 2d_f/d_s, \tag{8.4}$$

and

$$t_F = m_F \rho_F.$$

The expressions m_F and l_F are the mass and length factors of the fractal F. ρ_F is the resistance factor, and we call t_F the time factor, as that is what comes in when scaling X_t. d_f is the fractal dimension (i.e., Hausdorff dimension) of F. d_s is called the spectral dimension, and we shall see later that it is the asymptotic density of states in the spectrum of the infinitesimal generator of X_t; see [1, 14, 22, 23, 24, 31, 32, 52, 72, 73, 76, 77, 78, 79, 82]. d_w is called the dimension of the walk, and we shall see later that it is related to the range of X_t. The inequality (8.2) shows $d_s < d_f$, and so $d_w > 2$.

It is more convenient to normalize $W_n(t)$ by $(8\rho_F/9)^n = (t_F/l_F^2)^n$ rather than α_n. So we let $X_n(t) = W_n((t_F/l_F^2)^n t)$, and as in Section 7 take a weak limit of the laws of $X_n(t)$ to get a strong Markov process on F. We then extend X_t to $\widetilde{F}$. By Brownian scaling, $W_n(t)$ has the same law as $3^{-1}W_{n-1}(9t)$. So $X_n(t)$ has the same law as $3^{-1}X_{n-1}((9t_F/l_F^2)t) = 3^{-1}X_{n-1}(t_F t)$. Taking limits, if X_t is a Brownian motion on F, so is $3^{-1}X(t_F t)$. (However, we do not know that the latter process has the same law as X_t; see Section 15.)

§9. Some estimates

d_s and d_w are defined in terms of ρ_F, which is in turn defined as a limit of some variational problems, and it does not appear that there is a closed form expression for d_s. However we can get some quite good estimates for $\sigma_r(x)$ in terms of d_s and d_w.

First we need the following lemma.

Lemma 9.1. *Suppose*

$$\mathbb{P}(Y_i \le x \mid Y_1, \ldots, Y_{i-1}) \le p + bx$$

for each i. Then

$$\mathbb{P}\Big(\sum_{i=1}^n Y_i \le x\Big) \le \exp\Big(2\Big(\frac{bnx}{p}\Big)^{1/2} - n\log(1/p)\Big).$$

Proof. The hypothesis implies

$$\mathbb{E}(e^{-uY_i} \mid Y_1, \ldots, Y_{i-1}) \le p + \int_0^{(1-p)/b} e^{-ux} b\, dx \le p + bu^{-1}.$$

Then

$$\mathbb{P}\Big(\sum Y_i \le x\Big) = \mathbb{P}\Big(e^{-u\sum Y_i} \ge e^{-ux}\Big) \le e^{ux}\mathbb{E}e^{-u\sum Y_i}$$
$$\le e^{ux}(p+bu^{-1})^n$$
$$\le p^n \exp(ux + bn/(pu)).$$

Now take $u = (bn/px)^{1/2}$. □

Using Proposition 6.4 and taking limits, we see there exists $c_1 < 1$ such that

$$\mathbb{P}^x(\tau(X) \le s) \le c_1 + c_2 s, \qquad x \in F_n \cap [0,1/2]^2. \tag{9.1}$$

We combine (9.1) and Lemma 9.1 to obtain

Proposition 9.2. *There exist c_1 and c_2 such that if $x \in F$,*

$$\mathbb{P}^x(\sigma_r(x) \le t) \le c_1 \exp(-c_2(t_F^r t)^{-1/(d_w-1)}).$$

Proof. By scaling, it is enough to consider the case $r = 0$. To exit $D_0(x)$, the process must first exit at least $3^m/2$ blocks of the form $D_m(z)$. By (9.1) and scaling together with Lemma 9.1 and the strong Markov property,

$$\mathbb{P}^x(\sigma_0(x) \le s) \le \exp\big(c_3((3t_F)^m s)^{1/2} - (3^m/2)\log(1/c_4)\big).$$

Now set $m = [c_5 \log(c_6/s)]$ for the appropriate constants c_5 and c_6. □

An immediate corollary of Proposition 9.2 is

Corollary 9.3. *There exist c_1 and c_2 such that*

$$\mathbb{P}^x(\sup_{s\le t} |X_s - X_0| > \lambda) \le c_1 \exp\big(-c_2(\lambda^{d_w}/t)^{1/(d_w-1)}\big).$$

§10. Transition densities – upper bounds

Now that we have constructed our process, we would like to be able to discuss some of its properties. To do that, we proceed to obtain estimates on the transition densities.

It is natural to define the Laplacian on $\widetilde{F}$ to be twice the infinitesimal generator of X_t. The heat equation on $\widetilde{F}$ is

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}\Delta u(x,t).$$

The transition densities $p(t,x,y)$ are then the fundamental solution to the heat equation.

There is a well-known procedure for obtaining upper bounds on the fundamental solution to the heat equation from Nash inequalities (and in higher dimensions, from Sobolev inequalities); see [19]. The Nash inequality is obtained from geometric considerations. None of that can work here because, as we shall see, the parameters must involve d_s, which does not have a nice geometric interpretation. So new methods are needed.

We sketch a procedure for obtaining an upper bound for $p(t,x,x)$, the transition density of X_t with respect to the Hausdorff-Besicovitch measure μ. Throughout this section we consider our process to have state space $\widetilde{F}$ rather than just F.

Let C be a small set containing x. We then write

$$\mathbb{E}^x \int_0^\infty e^{-\lambda t} 1_C(X_t)\, dt = \sum_{r=-\infty}^{\infty} \mathbb{E}^x \int_{\sigma_{2r+2}(x)}^{\sigma_{2r}(x)} e^{-\lambda t} 1_C(X_t)\, dt. \tag{10.1}$$

By the strong Markov property, each summand can be written as

$$\mathbb{E}^x \Big[e^{-\lambda \sigma_{2r+2}(x)} \mathbb{E}^{X(\sigma_{2r+2}(x))} \int_0^{\sigma_{2r}(x)} e^{-\lambda t} 1_C(X_t)\, dt \Big].$$

The terms for r very negative correspond to $D_{2r}(x)$ being very large, hence $\sigma_{2r+2}(x)$ being very large. The expression $\mathbb{E}^x e^{-\lambda\sigma_{2r+2}(x)}$ implies that the contributions of these terms is small. When r is very positive, then $\sigma_{2r}(x)$ is very small, and X_t cannot spend much time in C. The dominant term in the series (10.1) occurs for r such that $\lambda \approx c_1 t_F^{2r}$, for then $\mathbb{E}^x e^{-\lambda\sigma_{2r+2}(x)} \approx 1$; (see Proposition 9.2). The dominant term is then roughly of the form

$$\mathbb{E}^x \mathbb{E}^{X(\sigma_{2r+2}(x))} \int_0^{\sigma_{2r}(x)} 1_C(X_t)\, dt.$$

By scaling, it will do to obtain a bound on

$$\mathbb{E}^y \int_0^{\sigma_0(x)} 1_C(X_t)\, dt, \qquad y \in \partial D_2(x).$$

If X_t killed on exiting $D_0(x)$ had a Green function $u(\cdot,\cdot)$ and C is small enough, then the above expression is roughly $u(y,x)\mu(C)$. By the Harnack inequality, $u(y,x)$ is comparable to $u(y,z)$, $z \in D_4(x)$, so

$$\begin{aligned} u(y,x) &\le c_2 \frac{1}{\mu(D_4(x))} \int_{D_4(x)} u(y,z)\,\mu(dz) \\ &\le c_2 \frac{1}{\mu(D_4(x))} \int_{D_0(x)} u(y,z)\,\mu(dz) \\ &\le c_2 \frac{1}{\mu(D_4(x))} \mathbb{E}^y \sigma_0(x). \end{aligned}$$

Since $\mu(D_4(x)) \ge c_3$ and $\mathbb{E}^y\sigma_0(x) \le \mathbb{E}^y \sigma_{-2}(y) \le c_4$, we obtain an upper bound of c_2c_4/c_3 for $u(y,x)$. Of course, we do not know that a Green function exists, but if we apply the above argument to the approximating processes $X_n(t)$ and pass to the limit, we obtain the existence of $u(y,x)$ for $y \ne x$ and we obtain bounds on its size.

Carrying out the above argument and paying attention to the constants, we end up with the estimate

$$\mathbb{E}^x \int_0^\infty e^{-\lambda t} 1_C(X_t)\,dt \le c_5 \mu(C) \lambda^{d_s/2-1}.$$

Consequently, if we set

$$u^\lambda(x,y) = \int_0^\infty e^{-\lambda t} p(t,x,y)\,dt, \tag{10.2}$$

we obtain

$$u^\lambda(x,x) \le c_5 \lambda^{d_s/2-1}.$$

The function $p(t,x,x)$ is decreasing in t for each x (this is a general fact about symmetric processes; it can also be seen from the eigenvalue expansion in Section 11). So, taking $\lambda = 1/t$,

$$tp(t,x,x) \le e \int_0^t e^{-1} p(s,x,x)\,ds \le e u^\lambda(x,x),$$

from which we obtain

$$p(t,x,x) \le c_6 t^{-d_s/2}. \tag{10.3}$$

From general considerations about symmetric processes (or from the eigenvalue expansion), $p(t,x,y) \le p(t,x,x)^{1/2} p(t,y,y)^{1/2}$, and so we conclude

Proposition 10.1. *There exists c_1 such that*

$$p(t,x,y) \le c_1 t^{-d_s/2}.$$

A much better bound is available when $x \ne y$.

Theorem 10.2. *There exist c_1 and c_2 such that*

$$p(t,x,y) \le c_1 t^{-d_s/2} \exp(-c_2(|x-y|^{d_w}/t)^{1/(d_w-1)}), \qquad x,y \in \widetilde{F}.$$

If d_f were equal to d_s, which is not the case for the Sierpinski carpet we treat in this section, then d_w would be 2, and this would reduce to the Gaussian tail.

Proof. We combine the global upper bound of Proposition 10.1 with the large deviations estimate of Proposition 9.2. Let

$$A = \{z : |z-x| \le |z-y|\}$$

and

$$S = \inf\{t : |X_t - X_0| > |x-y|/3\}.$$

Then

$$\mathbb{P}^x(X_t \in dy) = \mathbb{P}^x(X_t \in dy, X_{t/2} \notin A) + \mathbb{P}^x(X_t \in dy, X_{t/2} \in A). \tag{10.4}$$

By the Markov property and Proposition 10.1, the first term is

$$\begin{aligned}\mathbb{E}^x\Big[\mathbb{P}^{X_{t/2}}(X_{t/2} \in dy); X_{t/2} \in A^c\Big] &\le c_3 t^{-d_s/2}\mathbb{P}^x(X_{t/2} \in A^c)\\ &\le c_3 t^{-d_s/2}\mathbb{P}^x(S < t/2)\\ &\le c_3 t^{-d_s/2}\exp(-c_4(|y-x|^{d_w}/t)^{1/(d_w-1)}).\end{aligned}$$

The symmetry of $p(t,x,y)$ shows that the second term in (10.4) is equal to $\mathbb{P}^y(X_t \in dx, X_{t/2} \in A)$, and this can then be bounded in the same way that we bounded the first term of (10.4). □

§11. Transition densities – lower bounds

We turn now to lower bounds for $p(t,x,y)$. We first use the large deviations result (Proposition 9.2) to show that $p(t,x,x)$ cannot be too small on the diagonal.

Proposition 11.1. *There exists c_1 such that*

$$p(t,x,x) \ge c_1 t^{-d_s/2}, \qquad x \in \widetilde{F}.$$

Proof. Choose r to be the smallest integer such that

$$\mathbb{P}^x(X_{t/2} \in D_r(x)) \ge \mathbb{P}^x(\sigma_r(x) > t/2) \ge 1/2.$$

Thus r is chosen so that $t_F^r t$ is comparable to 1. Note

$$\mu(D_r(x)) \leq 4(8^{-r}) \leq c_2 t^{d_s/2}.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} 1/4 \leq (\mathbb{P}^x(X_{t/2} \in D_r(x)))^2 &= \Big(\int_{D_r(x)} p(t/2,x,y)\,\mu(dy) \Big)^2 \\ &\leq \mu(D_r(x)) \int_{D_r(x)} p(t/2,x,y)^2\,\mu(dy). \end{aligned}$$

By the semigroup property,

$$p(t,x,x) = \int p(t/2,x,y)p(t/2,x,y)\,\mu(dy) \geq \int_{D_r(x)} p(t/2,x,y)^2\,\mu(dy),$$

and combining proves the proposition. □

We now want to show that $p(t,x,y)$ is larger than some constant multiple of $t^{-d_s/2}$ if y is sufficiently close to x. The idea is to show that $p(t,x,y)$ is Hölder continuous in y by showing that $p(t,x,y)$ is the potential of a bounded function.

More precisely, let m be fixed and let $\overline{p}(t,x,y)$ be the transition densities for X_t killed on exiting $D_m(x_0)$. We want to show that $\overline{p}(t,x,y)$ is Hölder continuous in y with a modulus of continuity independent of m; we then let $m \to -\infty$. By the Hilbert-Schmidt theorem (see [16], Sect. II.4), $\overline{p}(t,x,y)$ has an eigenvalue expansion:

$$\overline{p}(t,x,y) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \varphi_i(x)\varphi_i(y).$$

Note that a consequence of the eigenvalue expansion is that $\overline{p}(t,x,x)$ is decreasing in t for each x; so the same must be true for $p(t,x,x)$. By the eigenvalue expansion and the Cauchy-Schwarz inequality, $\overline{p}(t,x,y) \leq \overline{p}(t,x,x)^{1/2}\overline{p}(t,y,y)^{1/2}$; again the same will then be true with $\overline{p}(t,x,y)$ replaced by $p(t,x,y)$.

Fix t and β and let

$$R(y) = \sum_{i=1}^{\infty} (\beta + \lambda_i) e^{-\lambda_i t} \varphi_i(x)\varphi_i(y).$$

It follows that $\overline{U}^\beta R(y) = \overline{p}(t,x,y)$, where $\overline{U}^\beta$ denotes the β-resolvent for X_t killed on exiting $D_m(x_0)$. Since $\sup_{\lambda\geq 0}(\beta+\lambda)e^{-\lambda t/2}$ is bounded by a constant c_1 depending only on β and t, then

$$\begin{aligned}\sum(\beta+\lambda_i)e^{-\lambda_i t}\varphi_i(z)^2 &\leq c_1 \sum e^{-\lambda_i t/2}\varphi_i(z)^2\\ &= c_1\overline{p}(t/2,z,z) \leq c_2 t^{-d_s/2}.\end{aligned}$$

This and Cauchy-Schwarz implies that $|R(z)|$ is bounded independently of m. By Proposition 7.2 and its proof, $\overline{U}^\beta R(y)$ is Hölder continuous in y with a modulus depending only on $\|R\|_\infty$ and β, which as we have seen, is independent of m.

Let c_3 be the constant c_1 in the statement of Proposition 11.1. By the above and scaling, there exists c_4 such that

$$|p(t,x,x) - p(t,x,y)| \leq c_3 t^{-d_s/2}/2$$

if $|x-y| \leq c_4 t^{1/d_w}$. Combining with Proposition 11.1 we have

Proposition 11.2. *There exist c_1 and c_2 such that*

$$p(t,x,y) \geq c_1 t^{-d_s/2}$$

if $|x-y| \leq c_2 t^{1/d_w}$.

We now use what is known as a chaining argument (cf. [20]) to get the off-diagonal lower bound.

Theorem 11.3. *There exist c_1 and c_2 such that*

$$p(t,x,y) \geq c_1 t^{-d_s/2} \exp(-c_2|x-y|^{d_w}/t)^{1/(d_w-1)}, \qquad x,y \in \widetilde{F}.$$

The idea here is that the main contribution to the lower bound is from those paths that go more or less directly from x to y.

Proof. Let $D = |x-y|$. For the appropriate constant c_3 the result will follow by Proposition 11.2 if $D \leq c_3 t^{1/d_w}$. So let us suppose $D > c_3 t^{1/d_w}$. Let n be the largest integer less than $c_4 t^{-1/(d_w-1)} D^{d_w/(d_w-1)}$ for an appropriate constant c_4. Let $x_0 = x, x_n = y$, and select points $x_1,\ldots,x_{n-1} \in \widetilde{F}$ such that $|x_{i+1}-x_i| \leq 2D/n$. Let $B_i = B(x_i, D/n) \cap \widetilde{F}$. Then by the semigroup property,

$$p(t,x,y) \geq \int_{B_1}\cdots\int_{B_{n-1}} p\Big(\frac{t}{n},x,y_1\Big)\cdots p\Big(\frac{t}{n},y_{n-2},y_{n-1}\Big)p\Big(\frac{t}{n},y_{n-1},y\Big)\,\mu(dy_1)\cdots\mu(dy_{n-1}).$$

Our choice of n guarantees that Proposition 11.2 applies, so $p(t/n, y_{i-1}, y_i) \geq c_5(t/n)^{-d_s/2}$. Therefore

$$p(t,x,y) \geq \prod_{i=1}^{n-1} \mu(B_i) c_5^n (t/n)^{-d_s n/2}.$$

Substituting and some algebra yields the theorem. □

§12. Path properties

Moduli of continuity. It is rather straightforward to use the estimates we have developed to obtain a modulus of continuity and a law of the iterated logarithm for the paths of X_t.

Transience and recurrence. The lower bound for $p(t,x,y)$ shows that $\int_0^\infty p(t,x,y)\,dt = \infty$ for all x and y. This implies that X_t is neighborhood recurrent; see [18].

If x_0 is fixed and u_0 is the Green function for X_t killed on exiting $D_0(x_0)$, then $u_0(x_0,x)$ is bounded, $u_0(x_0,y)=0$ on the boundary of $D_0(x_0)$, and $u_0(x_0,y)>0$ on $\partial D_2(x_0)$. Since $u_0(x_0,X_t)$ is a martingale up until time $\sigma_0(x_0)$, it follows that there exists c_1 such that starting at any $y \in \partial D_2(x_0)$, we have

$$\mathbb{P}^y(T_{\{x_0\}} < \sigma_0(x_0)) \geq c_1.$$

This, the fact that X_t returns to $D_2(x_0)$ infinitely often, and a renewal argument implies X_t hits x_0 infinitely often, or X_t is point recurrent.

One can also show that each point is regular for itself, i.e.,

$$\mathbb{P}^x(T_{\{x\}} = 0) = 1.$$

Local times. Let $\overline{u}$ be the Green function for X_t killed on exiting F. So $\overline{u}(X_t)$ is a supermartingale up to time τ. Our estimates imply that $\overline{u}$ is bounded and a little bit more work shows that $\overline{u}$ is jointly Hölder continuous. Since $\overline{u}$ is bounded, $\overline{u}(X_t,x)$ is a bounded supermartingale, and by the Doob-Meyer decomposition, there exists an increasing process $L(t,x)$ such that $\overline{u}(X_t,x)+L(t,x)$ is a martingale. $L(t,x)$ is the local time of $X_{t\wedge\tau}$ at x, and standard arguments shows that $L(t,x)$ increases only when X_t is at x and that $\int f(x)L(t,x)\mu(dy) = \int_0^{t\wedge\tau} f(X_t)\,dt$ if f is nonnegative and bounded. The Hölder continuity of $\overline{u}$ can be used to show that $L(t,x)$ is jointly continuous in t and x, a.s.

Spectral dimension. Recall the eigenvalue expansion for $\overline{p}(t,x,y)$, the transition density of X_t killed on exiting some $D_m(x_0)$. The form of the transition densities and standard arguments can be used to prove

$$\lim_{\lambda\to\infty}\frac{\log N(\lambda)}{\log\lambda}=d_s/2, \tag{12.1}$$

where $N(\lambda)$ is the number of eigenvalues λ_j that are less than or equal to λ. This is the justification for calling d_s the spectral dimension.

§13. Other fractals

Suppose instead of the standard Sierpinski carpet we consider fractals that are constructed in a similar way. We start with $F_0=[0,1]^2$, divide F_0 into l_F^2 equal subsquares, and remove a symmetric pattern of the subsquares to obtain F_1. Let m_F be the number of subsquares that remain. Now take each of the subsquares that make up F_1 and repeat the process: divide into l_F^2 equal parts and remove the same symmetric pattern of squares as was done in forming F_1 from F_0. Continuing, let F_n be the nth stage and let $F=\cap_{n=0}^{\infty}F_n$.

All of the above construction and estimates continue to hold provided F_1 satisfies four conditions. The first condition is that F_1 be symmetric with respect to all the isometries of the square $[0,1]^2$. This condition appears to be absolutely crucial; without it we have no idea how to proceed.

The second condition is one of connectedness. We require that the set that consists of the union of the interiors of the squares making up F_1 be a connected set. This clearly is necessary, or else the Brownian motion $W_1(t)$ cannot proceed from one side of F_1 to the other. The reason for looking at the interior of F_1 is that a Brownian motion cannot pass from one subsquare to another through a corner.

Next we require a local connectedness property. This means that if we take any square S of side length $2l_F$ contained in $[0,1]^2$ that consists of four of the l_F^2 subsquares of F_0, then either $S\cap F_1$ is empty or $S\cap F_1$ has connected interior. This is to avoid the situation where we have squares occupying the first and third quadrants of S but not the second and fourth. This hypothesis seems less crucial. A Brownian motion can pass from the first quadrant to the third quadrant of S because the interior of F_1 is connected, but it cannot do so by passing through the center. We expect that in the limit, the center of each such square S should be considered as two separate points, somewhat analogously to the Ray topology.

The final condition we impose on F_1 is that the all the subsquares of F_0 that touch the x axis must be contained in F_1. This is imposed in order that the distance from a point x to y within F be comparable to the Euclidean distance. By changing the metric on F suitably, this condition could be relaxed; see [8].

For all the fractals satisfying the above four conditions, the spectral dimension d_s is less than the fractal dimension, which is obviously less than two.

Suppose we consider fractals formed in a similar way, but now as subsets of $[0,1]^d$, $d \geq 3$. The property that it is possible to encircle a point by a curve no longer holds, and so our proof of the Harnack inequality no longer works. Kusuoka and Zhou [58] showed how to construct a Brownian motion for fractals embedded in $\mathbb{R}^d$ provided $d_s < 2$.

In [9] a method was announced for proving the uniform Harnack inequality for fractals satisfying the analogues of the four above conditions that live in $[0,1]^d$, and complete proofs are provided in [10]. The idea is to use coupling.

In a manner analogously to that in Section 3, define $D_n(x)$ to be a block of 2^d cubes such that x is near the center of $D_n(x)$. Let us say that $x \overset{n}{\sim} y$ if there exists cubes S_x, S_y of side length l_F^{-n} contained in F_n with $x \in S_x, y \in S_y$ and there exists an isometry taking S_x to S_y so that x gets mapped onto y. We construct, by means of suitable reflections, two Brownian motions W_t^x started at x and W_t^y started at y (by no means independent) such that upon exiting $D_n(x)$ and $D_n(y)$, respectively, there is positive probability at least $c_1 > 0$ that $W^x(\sigma_n(x)) \overset{n-1}{\sim} W^y(\sigma_n(y))$ and it is certain that $W^x(\sigma_n(x)) \overset{n}{\sim} W^y(\sigma_n(y))$. Let σ_n^r be the rth time the W processes exit a block of the form $D_n(z)$ for some z. By repeating the process, there is probability at least $1-(1-c_1)^r$ that

$$W^x(\sigma_n^r) \overset{n-1}{\sim} W^y(\sigma_n^r).$$

Choosing $r = r(n)$ suitably, and using induction, we find that there is positive probability that $W^x(S) \overset{0}{\sim} W^y(S)$, where S is the first time either process moves more than a certain distance from its starting point. What makes things work here is that the probability that $W^x(S) \overset{0}{\sim} W^y(S)$ is greater than

$$1 - \sum_{i=0}^{n} (1-c_i)^{r(i)},$$

while the maximum distance each process has moved is of the order

$$\sum_{i=0}^{n} r(i) l_F^{-i}.$$

Once we have $W^x(S) \overset{0}{\sim} W^y(S)$, it is an easy matter to see that W^x and W^y can meet, or couple, with positive probability.

Coupling implies the Harnack inequality. Suppose n is fixed, h is nonnegative and harmonic on F_n, and $x, y \in F_n \cap [0,r]^d$ for some r small. Let W^x and W^y be constructed so that if T is the time of coupling and S is the time that either W^x or W^y hits $\partial_a F$, then the probability of $\{T \geq S\}$ is less than some number $\rho < 1$. (It may be necessary to take r small to ensure this, but this can be done independently of n.) If $\mathbb{P}^{x,y}$ represents the joint law, then

$$h(x) = \mathbb{E}^{x,y}[h(W^x(T); T < S] + \mathbb{E}^{x,y}[h(W^x(S); T \geq S]$$

by optional stopping, and a similar equation holds with x replaced by y. Note that $W^x(T) = W^y(T)$, since T is the time of coupling. Taking the difference of the two equations and recalling that $h \geq 0$, we obtain

$$h(x) - h(y) \leq \mathbb{E}^x[h(W^x(S)); T \geq S] \leq \|h\|_\infty \mathbb{P}^{x,y}(T \geq S).$$

Using symmetry, we have

$$|h(x) - h(y)| \leq \rho \|h\|_\infty. \tag{13.1}$$

The Harnack inequality is a consequence of (13.1) together with scaling and the fact that a process $W_n(t)$ can be shown to hit cubes (by the analogues of knight and corner moves). See [17], Th. 3.9, for an example of how (13.1) is used.

The analogues of the resistance estimates still hold; see [10]. Transition density estimates may also be obtained, with differences, however, in the proofs of the on-diagonal upper bound and near-diagonal lower bound.

For the on-diagonal upper bound, one would not expect $\int_0^\infty e^{-\lambda t} p(t,x,x)\,dt$ to be finite. Instead one looks at

$$\int_0^\infty e^{-\lambda t} t^p p(t,x,x)\,dt$$

for $p > d$; with suitable modifications the argument of Section 11 goes through.

The near-diagonal lower bound (Proposition 11.2) may be obtained in the same way as for the Sierpinski carpet. However, the coupling allows a shorter proof; see [10].

§14. Further results

The proofs of the following may be found in [10]. They are, however, similar to proofs that may be found in the literature.

Transience and recurrence. Consider X_t on a fractal $\widetilde{F}$ constructed as in the previous section. If $d_s < 2$, then the process will be point recurrent, and jointly continuous local times exist. Points are regular for themselves, that is, starting at x, the hitting time $T_{\{x\}} = \inf\{t > 0 : X_t = x\}$ will be 0 with $\mathbb{P}^x$ probability one. If $d_s = 2$, then the process will be neighborhood recurrent but not point recurrent. It is not clear that there are any fractals with $d_s = 2$ other than standard two-dimensional Brownian motion. If $d_s > 2$, then the process is transient.

Rate of escape. When $d_s > 2$, the process is transient as we just said. One can give a rate of escape to infinity for the process. With probability one

$$\liminf_{t\to\infty} \frac{|X_t|}{t^{1/d_w}(\log t)^\gamma}$$

is zero if $\gamma > 1/(d_w - d_f)$ and infinity if $\gamma < 1/(d_w - d_f)$.

Self-intersections. Self-intersections exist if and only if $d_s < 4$. This means that the path of X_t will hit itself only if the spectral dimension is less than four.

Hausdorff dimension of the range. Suppose $d_s > 2$. Then the Hausdorff dimension of the range of X_t, i.e., the Hausdorff dimension of the set $\{X_t(\omega) : t \in [0, 1]\}$, will be equal to d_w almost surely. This provides some justification for calling d_w the dimension of the walk.

Sobolev inequalities. From our transition density estimates one can obtain the analogues of some classical inequalities; see [19]. For example, we consider the Sobolev inequality. Let $\mathcal{E}(f, f)$ denote the Dirichlet form for the process X_t on $\widetilde{F}$, and suppose $d_s > 2$. If $p = 2d_s/(d_s - 2)$, then we have

$$\|f\|_p^2 \le c_1 \mathcal{E}(f, f),$$

where c_1 does not depend on f. An analogue of the Poincaré inequality also exists.

Pre-carpet. Recall the definition of the pre-carpet $\widetilde{F}_0$ from (3.3). One can obtain good transition density estimates for a Brownian motion W_t that has normal reflection on $\partial\widetilde{F}_0$. Here the interest is that the transition density estimates have different forms depending on whether $|x-y|$ is larger than or less than t.

Graphical Sierpinski carpet. Let G be the graph whose vertices are the centers of the unit squares that are contained in the pre-carpet $\widetilde{F}_0$. Two points x and y are connected by an edge if $|x-y|=1$. The symmetric random walk on G will be transient if $d_s>2$ and recurrent if $d_s\le 2$. An analogue of the Sobolev inequality also holds.

It is interesting to note that in the inequalities and in many of the path properties, the spectral dimension plays at least as important a role as the fractal dimension.

§15. Open problems

Uniqueness. We showed in Section 7 that the sequence $X_n(t)$ had subsequential limit points with respect to the topology of weak convergence. We did not show that the full sequence converged. Had we shown that, then it would follow easily that the limit process was self-similar.

More generally, how does one know that some other construction of a Brownian motion on the Sierpinski carpet leads to the same process? For example, in [58] random walk approximations are used. Does the limit in this case have the same limit as the process constructed in Section 7? Another possibility is to start with $W_n(t)$ being Brownian motion in F_n conditioned not to hit $\partial_r F_n$ rather than being reflecting Brownian motion. Would one have the same limit process?

Transition density estimates for the Kusuoka-Zhou process. The transition density estimates we obtained in Sections 10 and 11 apply to any of the processes constructed in Section 7. Do they also apply to the self-similar processes constructed by Kusuoka and Zhou?

Divergence form operators. Divergence form operators in $\mathbb{R}^d$ are operators corresponding to the Dirichlet form

$$\mathcal{E}(f,f) = 1/2 \int_{\mathbb{R}^d} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x)\, dx. \tag{15.1}$$

Can one construct processes with state space $\widetilde{F}$ that have the same relationship to Brownian motion on the Sierpinski carpet as the processes corresponding to (15.1) do to Brownian motion on $\mathbb{R}^d$?

Oscillations in the density of states. Fukushima and Shima [23, 24] showed that

$$\liminf_{\lambda\to\infty} \lambda^{-d_s/2} N(\lambda) < \limsup_{\lambda\to\infty} \lambda^{-d_s/2} N(\lambda),$$

where $N(\lambda)$ is the number of eigenvalues for Brownian motion on the Sierpinski gasket less than λ; see Section 12. (See also [14]). Is the analogous property true for the eigenvalues of Brownian motion on the Sierpinski carpet?

Random Sierpinski carpets. Suppose we have two patterns F_1 and F_1'. At each stage in the construction of the fractal, suppose we replace each subsquare by one of the patterns F_1, F_1'; which one we choose is determined by some random mechanism (cf. [12]). Can one construct a Brownian motion on the resulting fractal and determine estimates for the transition densities?

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Richard F. BASS *Department of Mathematics, University of Washington, Seattle, Washington 98195-4350, USA.* E-mail:bass@math.washington.edu