ERRATA: Diffusions and Elliptic Operators

Updated August 19, 2008

Page 17, line 1: The integral should be

$$\int_0^t g_n'(|X_s - X_s'|)\operatorname{sgn}(X_s - X_s')[b(X_s) - b(X_s')]\,ds$$

Page 19, line 4: $\nu \geq 2$

Page 19, lines 6,7: Insert between lines 6 and 7: We can also define Bessel processes of order $\nu \in [0,2)$ by (7.3) for t up to the first hit of 0; at 0 a local time term needs to be included.

Page 21, line 2: and $X_t \to \infty$ a.s.

Page 21, line 9: ... is a martingale up to the first hit of 0.

Page 24, lines 5-15: where $|e_s|$ is bounded, say by c_2 (cf. PTA, theorem I.5.11]). Let $\tau = \inf\{t > 0 : |X^1_{B_t}| > N\}$. Since

$$|X^1_{B_{n+1}} - X^1_{B_n}| \geq |\widetilde{W}_{n+1} - \widetilde{W}_n| - c_2,$$

then $|X^1_{B_{n+1}}| > N$ on the set $\{|X^1_{B_n}| < N, |\widetilde{W}_{n+1} - \widetilde{W}_n| > c_2 + 2N\}$. There exists $c_3 \in (0,1)$ such that $\mathbb{P}(|\widetilde{W}_{n+1} - \widetilde{W}_n| > c_2 + 2N) > c_3$. By the independent increment property of Brownian motion,

$$\begin{aligned}\mathbb{P}(\tau > n+1) &= \mathbb{P}(\tau > n+1, \tau > n) = \mathbb{E}\left[\mathbb{P}(\tau > n+1 \mid \mathcal{F}_n); \tau > n\right] \\ &\leq (1 - c_3)\mathbb{P}(\tau > n).\end{aligned}$$

By induction, $\mathbb{P}(\tau > n) \leq (1 - c_3)^n$; hence $\tau < \infty$ a.s. Since $d\langle M\rangle_t/dt$ is

Page 25, lines 11-12:

Theorem 8.4. *Let W_t be a d-dimensional Brownian motion and let X_t be a d-dimensional process such that*

$$X_t^i = x_0 + \int_0^t \sum_{j=1}^d H_{ij}(s)\,dW_s^j + \int_0^t B_i(s)\,ds, \qquad i = 1, \ldots, d,$$

where H_{ij} and B_i are predictable and bounded. For each s and ω let $K(s)$ be the matrix that is the inverse of $H(s)$. Suppose there exists M such that *for all s, i, and j, $|H_{ij}(s)|, |K_{ij}(s)|$, and $|B_i(s)|$ are bounded by M a.s. Let $\varepsilon > 0$, $t_0 > 0$. There exists c_1 depending only on M such that*

Page 44, line 7: $\Lambda(x) > 0$

Page 44, Display (1.2): $\sum_{i=1}^d y_i^2$

Page 45, line -3: $e^{-\lambda s}\lambda u(X_s)\,ds$

Page 55, line 9: Put integral sign before P_t

Page 61, line 3 to Page 62, line 5: Replace with the following.

Let us say $f \in C^{2+\alpha}(D)$ if f, $\partial_i f$ and $\partial_{ij} f$ are bounded in D for all i, j and

$$\|f\|_{C^{2+\alpha}(D)} = \sup_{x\in D} |f(x)| + \sup_i \sup_{x\in D} |\partial_i f(x)| + sup_{i,j} \|\partial_{ij} f\|_{C^\alpha(D)}$$

is finite. We define $C^{1+\alpha}$ analogously.

(2.2) Proposition. *(a) If $f \in C^\alpha$ on B, then $P_B f \in C^\alpha$ on B and there exists c_1 independent of f such that*

$$\|P_B f\|_{C^\alpha(B)} \le c_1 \|f\|_{C^\alpha}.$$

(b)If $f \in C^{2+\alpha}$ on B, then $P_B f \in C^{2+\alpha}$ on B and there exists c_2 independent of f such that

$$\|P_B f\|_{C^{2+\alpha}(B)} \le c_2 \|f\|_{C^{2+\alpha}}.$$

Proof. Clearly $|P_B f(x)| \le \|f\|_\infty$, and $P_B f$ in B depends only on the values of f on ∂B. By [PTA, Proposition II.1.3], $P_B f$ is C^∞ in B. Using this and rotational invariance it suffices to obtain an estimate on $|P_B f(y) - P_B f(x)|$ for $x, y \in B(e_1, 1/4)$, where e_i is the unit vector in the x_i direction.

Let us write $f = f_1 + f_2$, where the C^α norms of f_1, f_2 are bounded by a constant times the C^α norm of f, f_1 is supported in $B(e_1, 1/2)$ and f_2 is 0 in $B(e_1, 3/8)$.

If we use the explicit formula for the Poisson kernel in B (see [PTA, Theorem II.1.17], for example) and differentiate it, we deduce that

$$|\nabla P_B f_2(x)| \le c_3 \|f_2\|_\infty, \qquad x \in B(e_1, 1/4).$$

We are using here the fact that f_2 is zero in $B(e_1, 3/8)$. Therefore by the mean value theorem,

$$|P_B f_2(y) - P_B f_2(x)| \le c_3 \|f_2\|_\infty |y - x| \le c_4 \|f\|_\infty |y - x|^\alpha.$$

We thus only need to consider $P_B f_1$ in $B(e_1, 1/4)$. Let us map $B(0,1)$ to $B(e_d, 1)$ by a translation, map $B(e_d, 1)$ to $H_{1/2} = \{(x_1, \ldots, x_d) : x_d > 1/2\}$ by inversion through the unit sphere, that is, the map $x \to I(x) = x/|x|^2$, and finally map $H_{1/2}$ to $H = \{(x_1, \ldots, x_d) : x_d > 0\}$ by a translation. The composite map is nonsingular in $B(e_1, 1/4)$ and the inversion map has the property that if u is harmonic in a domain D, then $|x|^{2-d} u(I(x))$ is harmonic in $I(D)$ (see [PTA, Lemma II.1.18] for these facts). Since f_1 is supported in $B(e_1, 1/2)$, it suffices to show

$$\|P_H g\|_{C^\alpha(H)} \le c_5 \|g\|_{C^\alpha(H)}, \tag{2.3.1}$$

where $P_H g$ is the harmonic extension of g in H, that is, $P_H g(x) = \mathbb{E}^x g(W_{\tau_H})$, where W is a Brownian motion and τ_H is the first exit time of W from H.

Write $\widetilde{x} = (x_1, \ldots, x_{d-1})$ so that $x = (\widetilde{x}, x_d)$. Define $g_{\tilde{z}}(\widetilde{x}, x_d) = g(\widetilde{x} + \widetilde{z}, x_d)$. We have

$$\begin{aligned} |P_H g(\widetilde{x}, x_d) - P_H g(\widetilde{y}, x_d)| &= |P_H g_{\tilde{x}}(0, x_d) - P_H g_{\tilde{y}}(0, x_d)| \\ &= |P_H (g_{\tilde{x}} - g_{\tilde{y}})(0, x_d)| \\ &\le \|g_{\tilde{x}} - g_{\tilde{y}}\|_{L^\infty(\partial H)} \le \|g\|_{C^\alpha(H)} |\widetilde{x} - \widetilde{y}|^\alpha. \end{aligned} \tag{2.3.2}$$

By Stein [1], Proposition V.7,

$$|\partial_d P_H g(\widetilde{x}, t)| \le c_6 \|g\|_{C^\alpha(H)} t^{-1+\alpha}.$$

So if $y_d > x_d$,

$$\begin{aligned} |P_H g(\widetilde{x}, y_d) - P_H g(\widetilde{x}, x_d)| &= \Big| \int_{x_d}^{y_d} \partial_d (P_H g)(\widetilde{x}, t)\, dt \Big| \\ &\le c_6 \|g\|_{C^\alpha(H)} \int_{x_d}^{y_d} t^{-1+\alpha}\, dt \\ &= c_7 \|g\|_{C^\alpha(H)} (y_d^\alpha - x_d^\alpha) \\ &\le c_8 \|g\|_{C^\alpha(H)} (y_d - x_d)^\alpha. \end{aligned} \tag{2.3.3}$$

Combining (2.3.2) and (2.3.3) proves (2.3.1).

(b) We decompose $f = f_1 + f_2$ as in (a), and handle $P_B f_2$ similarly to what was done in (a). By the same transformations of the state space as in (a), it is enough to show

$$\|P_H g\|_{C^{2+\alpha}(H)} \le c_9 \|g\|_{C^{2+\alpha}(H)}. \tag{2.3.4}$$

We may handle partials with respect to $x_1, \ldots, x_{d-1}$ as in (2.3.2), so it suffices to consider $\partial_{id} P_H g$ for $i \ne d$ and $\partial_{dd} P_H g$. Since $P_H g$ is harmonic in H, then $\partial_{dd} P_H g = -\sum_{i=1}^{d-1} \partial_{ii} P_H g$, which can be handled as in (2.3.2), so we are left to consider $\partial_{id} P_H g$ with $i \ne d$. The operators ∂_i and P_H commute if $i \ne d$ by the argument in (2.3.2), and writing G for $\partial_i g$, it therefore suffices to show

$$\|P_H G\|_{C^{1+\alpha}} \le c_{10} \|G\|_{C^{1+\alpha}}. \tag{2.3.5}$$

Differences in the $x_1, \ldots, x_{d-1}$ directions are handled as in (2.3.2), so we need to look at differences in the x_d direction. By Stein [1], Proposition V.9,

$$|\partial_{dd} P_H G(\widetilde{x}, t)| \le c_{11} \|G\|_{C^\beta} t^{-2+\beta},$$

where we take $\beta = 1 + \alpha$. Therefore

$$\begin{aligned} |\partial_d P_H G(\widetilde{x}, y_d) - \partial_d P_H G(\widetilde{x}, x_d)| &= \Big| \int_{x_d}^{y_d} \partial_{dd} P_H G(\widetilde{x}, t)\, dt \Big| \\ &\le c_{11} \|G\|_{C^{1+\alpha}(H)} \int_{x_d}^{y_d} t^{-1+\alpha}\, dt \\ &\le c_{12} \|G\|_{C^{1+\alpha}(H)} (y_d - x_d)^\alpha, \end{aligned}$$

similarly to (2.3.3). This is what we need to complete the proof of (b). □

Page 62, line -8:

$$\partial_{ij} G_{B(x_0,R)} f = \partial_{ij} U f - \partial_{ij} P_{B(x_0,R)} (Uf),$$

Page 77, line -2: > 0.

Page 160, lines 15–21: Replace by the following.

If $x, y \in Q$,

$$u(x) - u(y) = \int_0^{|y-x|} \partial_r u(y + rv) dr, \qquad v = (x-y)/|y-x|.$$

Integrating with respect to y,

$$|Q|[u(x) - u_Q] = \int_Q \int_0^{|y-x|} \partial_r u(x + rv) dr\, dy.$$

Set $V(z)$ equal to $|\nabla u(z)|$ if $z \in Q$ and 0 otherwise. Then

$$\begin{aligned} |u(x) - u_Q| &\le \frac{1}{|Q|} \int_{|y-x| \le 2\sqrt{d}} \int_0^\infty V(x + rv) dr\, dy \\ &\le c_2 \int_Q |y-x|^{1-d} V(y)\, dy. \end{aligned}$$

Now apply this inequality together with Theorem IV.5.1 of [PTA] where we set $p = 2$ and we set $K(x, y) = |y - x|^{1-d}$ if $x, y \in Q$ and 0 otherwise. We obtain

$$\int_Q |u(x)-u_Q|^2 dx \le c_2^2 \int_Q \Big[\int K(x,y)V(y)\,dy\Big]^2 dx \le c_3 \int_Q |\nabla u(x)|^2 dx.$$

Page 188. Replace the proof of Theorem 8.4 by the following.

Proof. As in the proof of Theorem 7.5, we may assume without loss of generality that $d \ge 3$. As in the proof of Theorem I.8.5, it suffices to consider the case where ψ is differentiable. By Proposition 6.7 there exists c_2 and c_3 such that if $|x-x_0| \le c_2 r^{1/2}$ and $|x-y| \le c_2 r^{1/2}$, then $p_{B(x_0,r^{1/2})}(x,y) \ge c_3 r^{-d/2}$. Choose n large so that if $r = t/n$, then $r^{1/2} \le \varepsilon/8$ and $r\|\psi'\|_\infty \le (c_2/2)r^{1/2}$. Let $y_i = \psi(ir)$. Let $c_4 = c_2/4$.

If $x \in B(y_i, c_4 r^{1/2})$ and $y \in B(y_{i+1}, c_4 r^{1/2})$, then

$$|x-y| \le 2c_4 r^{1/2} + |y_i - y_{i+1}| \le 2c_4 r^{1/2} + r\|\psi'\|_\infty \le c_2 r^{1/2}.$$

Taking $x_0 = y_i$, we see $p_{B(y_i,r^{1/2})}(x,y) \ge c_3 r^{-d/2}$. It follows that

$$\mathbb{P}^x(X_r \in B(y_{i+1}, r^{1/2}), \sup_{s\le r} |X_s - X_0| \le \varepsilon/4) \ge c_2 r^{-d/2} |B(y_i, r^{1/2})| \ge c_5.$$

Note

$$\begin{aligned}\mathbb{P}^{\psi(0)}&(\sup_{s\le t} |X_s - \psi(s)| < \varepsilon)\\ &\ge \mathbb{P}^{\psi(0)}(X_{ir} \in B(y_i, c_4 r^{1/2}), \sup_{s\le r} |X_s - X_{ir}| \le \varepsilon/4, i = 0, \dots, n),\end{aligned}$$

and applying the Markov property n times, this is greater than $c_5^n > 0$. □

Page 202, lines -10, -9: Replace by the following:

Since

$$\begin{aligned}0 = I - I &= V(W+\varepsilon H_j)V^{-1}(W+\varepsilon H_j) - V(W)V^{-1}(W)\\ &= (V(W+\varepsilon H_j) - V(W))V^{-1}(W)\\ &\quad + V(W+\varepsilon H_j)(V^{-1}(W+\varepsilon H_j) - V^{-1}(W)),\end{aligned}$$

then

$$V^{-1}(W+\varepsilon H_j) - V^{-1}(W) = -V^{-1}(W+\varepsilon H_j)(V(W+\varepsilon H_j) - V(W))V^{-1}(W).$$

Dividing both sides by ε and letting $\varepsilon \to 0$,