Errata for *Real Analysis for Graduate Students,*

Second Edition

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`http://www.math.uconn.edu/~bass/real.html`

The errata pages for the second edition will no longer be updated.

Page 4, lines 3,4: $B(x,r/2)$ and $B(y,r/2)$

Page 4, line -1: Add:
(3) if $x \le y$ and $y \le z$, then $x \le z$.

Page 11, Exercise 2.5: X into Y

Page 11, Exercise 2.6: whenever $A \in \mathcal{A}$ is non-empty,

Page 15, line -7: such that $(X, \overline{\mathcal{A}}, \overline{\mu})$ is complete, where $\overline{\mu}$ is a measure on $\overline{\mathcal{A}}$ that is an extension of μ, that is, $\overline{\mu}(B) = \mu(B)$ if $B \in \mathcal{A}$.

Page 16, line -1: Replace by the following:

$B \in \mathcal{B}$ if and only if there exists $A \in \mathcal{A}$ and $N \in \mathcal{N}$ such that $B = A \cup N$. Define $\overline{\mu}(B) = \mu(A)$ if $B = A \cup N$ with $A \in \mathcal{A}$ and $N \in \mathcal{N}$. Prove that $\overline{\mu}(B)$ is uniquely defined for each $B \in \mathcal{B}$, that $\overline{\mu}$ is a measure on $\mathcal{B}$, that $(X, \mathcal{B}, \overline{\mu})$ is complete, and that $(X, \mathcal{B}, \overline{\mu})$ is the completion of $(X, \mathcal{A}, \mu)$.

Page 20, line -7: that $\emptyset \in \mathcal{C}$ and there exist $D_1, D_2, \ldots$ in $\mathcal{C}$ such that $X = \cup_{i=1}^\infty D_i$. Suppose $\ell : \mathcal{C} \to [0, \infty]$ with

Page 21, line -4: $x \ge 0$

Page 25, lines 15–19: Replace by the following:

Let $\delta > 0$ and let A_j, $j = 1, 2, \ldots$, be elements of $\mathcal{C}$ such that $I_i \subset \cup_{j=1}^\infty A_j$ and

$$\sum_{j=1}^\infty \ell(A_j) \le m^*(I_i) + \delta.$$

Let $C_{ij} = I_i \cap A_j$, which will again be an interval (possibly empty) that is open on the left and closed on the right, and hence in $\mathcal{C}$. Write $J^c = K_1 \cup K_2$, where $K_1 = (-\infty, c]$ and $K_2 = (d, \infty)$.

Note $C_{ij} \cap J$ will be an interval that is open on the left and closed on the right, and the same is true of $C_{ij} \cap K_1$ and $C_{ij} \cap K_2$ (any of these could be empty). Using (4.4) twice,

$$\ell(C_{ij}) = \ell(C_{ij} \cap K_1) + \ell(C_{ij} \cap J) + \ell(C_{ij} \cap K_2).$$

We have that the set $I_i \cap J$ is contained in the union of the countable subcollection $\{C_{ij} \cap J\}_{j=1}^{\infty}$ of $\mathcal{C}$ and that the set $I_i \cap J^c$ is contained in the union of the countable subcollection $\{C_{ij} \cap K_1, C_{ij} \cap K_2\}_{j=1}^{\infty}$ of $\mathcal{C}$. Therefore

$$\begin{aligned}
m^*(I_i \cap J) &+ m^*(I_i \cap J^c) \\
&\le \sum_{j=1}^{\infty} \ell(C_{ij} \cap J) + \Big(\sum_{j=1}^{\infty} \ell(C_{ij} \cap K_1) + \sum_{j=1}^{\infty} \ell(C_{ij} \cap K_2) \Big) \\
&= \sum_{j=1}^{\infty} [\ell(C_{ij} \cap J) + \ell(C_{ij} \cap K_1) + \ell(C_{ij} \cap K_2)] \\
&= \sum_{j=1}^{\infty} \ell(C_{ij}) \le \sum_{j=1}^{\infty} \ell(A_j) \\
&\le m^*(I_i) + \delta.
\end{aligned}$$

Since δ is arbitrary,

$$m^*(I_i \cap J) + m^*(I_i \cap J^c) \le m^*(I_i).$$

Page 28, line -12: $\inf\{f_0(y) : y \ge x, y \notin C\}$

Page 29, line 22: finite measure

Page 32, line 11: every set in $\mathcal{A}_0$ and every μ^*-null set is μ^*-measurable;

Page 32, line -3: μ^*-measurable. That μ^*-null sets are μ^*-measurable follow by the definition of μ^*-measurable and the fact that μ^* satisfies Definition 4.1(2).

Page 33, line 4: $E \in \sigma(\mathcal{A}_0)$

Page 35, line 17: Add:

This is known as the *Steinhaus theorem*

Page 36, Exercise 4.16: Change to:

Exercise 4.16 (1) Give an example of a set X and a finite outer measure μ^* on X, subsets $A_n \uparrow A$ of X, and subsets $B_n \downarrow B$ of X such that $\mu^*(A_n)$ does not converge to $\mu^*(A)$ and $\mu^*(B_n)$ does not converge to $\mu^*(B)$.
(2) Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and define μ^* as in Exercise 4.3. Show that if $A_n \uparrow A$ for subsets A_n, A of X, then $\mu^*(A_n) \uparrow \mu^*(A)$.

Page 39, line 13: provided they are finite.

Page 39, line -13: $\{x : \sup_i f_i(x) > a\}$

Page 44, Exercise 5.6: $f : \mathbb{R} \to \mathbb{R}$

Page 49, line -1: $=$ instead of $\to$

Page 50, Exercise 6.7: Let $(X, \mathcal{A}, \mu)$ be a finite measure space

Page 58, line 12: Add:

For this problem you may use the fact that if f is continuous on $[a, b]$ and F is differentiable on $[a, b]$ with derivative f, then $\int_a^b f(x)\,dx = F(b) - F(a)$. This follows by the results of the next chapter and the fundamental theorem of calculus.

Page 76, between lines 15 and 16: Add:

If $A = \cap_{k=1}^\infty \cup_{j=k}^\infty A_j$, then $x \in A$ if and only if x is in infinitely many of the A_j. Sometimes one writes $A = \{A_j \text{ i.o.}\}$.

Page 83, line -1: $\mu(t_y(E_n))$

Page 85, line 10: Insert before "If either":

Suppose μ and ν are σ-finite measures on X and Y, resp.

Page 88, lines -1,-2: Change exponent from 3/2 to 3/4 in both lines

Page 93, line -10: Insert before "whenever":

with absolute convergence of the series when $\mu(\cup_{i=1}^\infty A_i)$ is finite

Page 96, lines 23, 24: $E = (\frac{1}{2}, 1]$ and $F = [0, \frac{1}{2}]$.

Page 98, Exercise 12.4: Move to just before Exercise 13.9 in the next chapter.

Page 98, Exercise 12.6: $|\mu + \nu|(A) \le |\mu|(A) + |\nu|(A)$

Page 105, Exercise 13.8: Suppose μ, ν, and ρ are finite measures, $\nu \ll \mu$, and $\rho \ll \nu$.

Page 114, line 6: for almost every x (with respect to the measure m)

Page 114, line -6: Change $\lambda((x,r))$ to $\lambda(B(x,r))$

Page 136, line 10: dense in $L^p(\mathbb{R})$ for $1 \le p < \infty$.

Page 143, Exercise 15.15: Let $p \in [1,\infty)$ and suppose μ is a finite measure.

Page 156, Exercise 16.6, line -2: completion of $\mathcal{L} \times \cdots \times \mathcal{L}$

Page 174, lines -3,-2: x_0 not in M such that $\inf_{x \in M} |x - x_0| > 0$, we can define $f(x + \lambda x_0) = \lambda$ for $x \in M$,

Page 175, line -3: and x_n both

Page 179, Exercise 18.3, line 5: times continuously differentiable

Page 180, Exercise 18.13: Let X be the space of continuously differentiable functions with the supremum norm and

Page 186, line -8: $+2\|y\|$

Page 187, line -11: Insert the following after line -11 and before line -10:

Lemma 19.8.1 Let M be a closed subspace of H with $M \neq H$. Then $M^\perp$ contains a non-zero element.

Proof. Choose $x \in H$ with $x \notin M$. Let $E = \{w - x : w \in M\}$. It is routine to check that E is a closed and convex subset of H. By Lemma 19.8, there exists an element $y \in E$ of smallest norm.

Note $y + x \in M$ and we conclude $y \neq 0$ because $x \notin M$.

We show $y \in M^\perp$ by showing that if $w \in M$, then $\langle w, y\rangle = 0$. This is obvious if $w = 0$, so assume $w \neq 0$. We know $y + x \in M$, so for any real number t we have $tw + (y + x) \in M$, and therefore $tw + y \in E$. Since y is the element of E of smallest norm,

$$\begin{aligned}\langle y, y\rangle = \|y\|^2 &\le \|tw + y\|^2 \\ &= \langle tw + y, tw + y\rangle \\ &= t^2\langle w, w\rangle + 2t\mathrm{Re}\,\langle w, y\rangle + \langle y, y\rangle,\end{aligned}$$

which implies

$$t^2\langle w, w\rangle + 2t\mathrm{Re}\,\langle w, y\rangle \ge 0$$

for each real number t. Choosing $t = -\mathrm{Re}\,\langle w,y\rangle/\langle w,w\rangle$, we obtain

$$-\frac{|\mathrm{Re}\,\langle w,y\rangle|^2}{\langle w,w\rangle} \geq 0,$$

from which we conclude $\mathrm{Re}\,\langle w,y\rangle = 0$.

Since $w \in M$, then $iw \in M$, and if we repeat the argument with w replaced by iw, then we get $\mathrm{Re}\,\langle iw,y\rangle = 0$, and so

$$\mathrm{Im}\,\langle w,y\rangle = -\mathrm{Re}\,(i\langle w,y\rangle) = -\mathrm{Re}\,\langle iw,y\rangle = 0.$$

Therefore $\langle w,y\rangle = 0$ as desired. □

If in the proof above we set $Px = y + x$ and $Qx = -y$, then $Px \in M$ and $Qx \in M^\perp$, and we can write $x = Px + Qx$. We call Px and Qx the *orthogonal projections* of x onto M and $M^\perp$, resp. It is an exercise to show that each element of H can be written as the sum of an element of M and an element of $M^\perp$ in exactly one way.

Page 191, lines 4,5: basis, then, is a subset of H

Page 194, Exercise 19.3: M is a closed subspace of

Page 194, Exercise 19.5: Remove exercise. (This is now Lemma 19.8.1.)

Page 201, line -3: that is dense in X.

Page 207, line 11: and by the

Page 216, line 6: $\leq 4\varepsilon$.

Page 224, line 3: $\{G_1, G_2, \ldots\}$

Page 231, line 6: $2\|g\|_\infty \int_{|y|>\delta} \varphi_\beta(y)\,dy$,

Page 232, line -2 through Page 233, line 16: Replace with the following:

Lemma 20.42. *Suppose $\mathcal{A}$ is an algebra of functions in $\mathcal{C}(X)$ such that $\mathcal{A}$ separates points and vanishes at no point. If x and y are two distinct points in X and a, b are two real numbers, there exists a function $f \in \mathcal{A}$ (depending on x, y, a, b) such that $f(x) = a$ and $f(y) = b$.*

Proof. Let g be a function in $\mathcal{A}$ such that $g(x) \neq g(y)$. Let h_x and h_y be functions in $\mathcal{A}$ such that $h_x(x) \neq 0$ and $h_y(y) \neq 0$. Define u and $v \in \mathcal{A}$ by

$$u(z) = g(z)h_x(z) - g(y)h_x(z)$$

and

$$v(z) = g(z)h_y(z) - g(x)h_y(z).$$

Note that $u(x) \neq 0$, $u(y) = 0$, $v(x) = 0$, and $v(y) \neq 0$. Now set

$$f(z) = \frac{a}{u(x)}u(z) + \frac{b}{v(y)}v(z).$$

This f is the desired function. □

Theorem 20.43. *Let X be a compact Hausdorff space and let $\mathcal{A}$ be a lattice of real-valued continuous functions with the property that whenever $x \neq y$ and $a, b \in \mathbb{R}$, then there exists $f \in \mathcal{A}$ (depending on $x, y, a,$ and b) such that $f(x) = a$ and $f(y) = b$. Then $\mathcal{A}$ is dense in $\mathcal{C}(X)$.*

Page 234, line 11: real-valued continuous functions

Page 234, line 18: in $\overline{\mathcal{A}}$

Page 245, Exercise 20.46: $P(z) = \sum_{j=0}^{n} \sum_{k=0}^{n}$

Page 249, line -2: $p \in (0, 1)$

Page 260, lines 8–9: $\mathbb{E}\,|X_1| < \infty$.

Page 266, line 11: $-S_n^2$

Page 274, line 1: support of g is contained in a closed interval

Page 274, lines 5–9: Replace with

$F_X(-M) < \varepsilon$ and so that M, $M+1$, $-M$, and $-M-1$ are continuity points of F_X and of the F_{X_n}. Let h be a bounded continuous functions that agrees with g on $[-M, M]$, has support contained in $[-M-1, M+1]$, and $\|h\|_\infty \leq \|g\|_\infty$. By the above argument, $\mathbb{E}\,h(X_n) \to \mathbb{E}\,h(X)$. The difference between $\mathbb{E}\,h(X)$ and $\mathbb{E}\,g(X)$ is bounded by

Page 297, line -8: Insert a comma between $a > 0$ and $M_n =$

Page 300, line 9: (2) Prove that if $a \in \mathbb{R}$, then

Page 304, line -4: $u\nabla v$,, where ∇v is the gradient of v, then

Page 315, Exercise 22.2, line 2: then uv is harmonic in D

Page 317, line -2: set of C^∞ functions

Page 321, line 15: Since $r > 1$, then $x \to |x|^r$ is continuously differentiable, and so $w \in C^1_K$. We observe

Page 334, line 4: to those of

Page 343, line 3: $\leq \frac{c_1}{\lambda^2} \|g\|_2^2$

Page 347, Exercise 24.9, line 3: $\lim_{\varepsilon \to 0, N \to \infty} \int_{\varepsilon < |x| < N} \frac{y_j}{|y|^{n+1}} f(x-y)\, dy$

Page 353, line -2: $\langle Ax, x \rangle$

Page 359, line -5: Suppose we have found eigenvectors $z_1, \ldots, z_n$ with corresponding eigenvectors $\lambda_1, \ldots, \lambda_n$.

Page 359, line -4: $Y = X_n^\perp$

Page 359, line -2: If $x \in Y$ and $k \leq n$, then

Page 370, line 4: $\leq 2|a|\, \|x\|\, \|Ax\| \leq$

Page 370, line 13: $\|(\lambda - A)(x_1 - x_2)\|^2$

Page 372, line 3: assertion (5) now

Page 372, line -2: Add at end:

Now use the right hand equality to define $L_{x,y} f$ for all f that are bounded and Borel measurable.

Page 375, line 5: $= \langle E(\sigma(A))x, y \rangle$.

Page 378, Exercise 25.9, line 1: n^{th} largest non-negative eigenvalue

Page 379, Exercise 25.10, line 1: n^{th} largest non-negative eigenvalue

Page 385, line 3: $\operatorname{supp}(f)$

Page 385, line 7: finitely many sets

Page 385, line -1: $\|f\|_{C^m(K)} \leq 1/m$

Page 388, line -13: there exist a non-negative integer L and

Page 389, line 16: $|D^j f(x) - D^j f(-x_0)|$

Page 389, line -4: only possible limit is equal to g a.e. Therefore we may assume that

Page 390, line 8: $= (-1)^k D^{2k} G_g(f)$

Page 391, line 14: $D^k \mathcal{F} f$ is a continuous function, and hence $\mathcal{F} f \in$

Page 395, item [2]: Bourdon, and W. Ramey.

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