Uniqueness of the Brownian motion on the Sierpinski carpet

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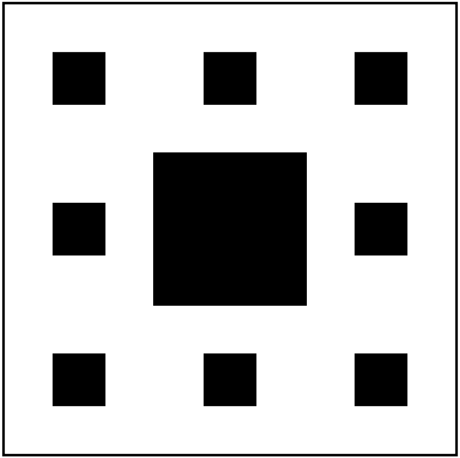
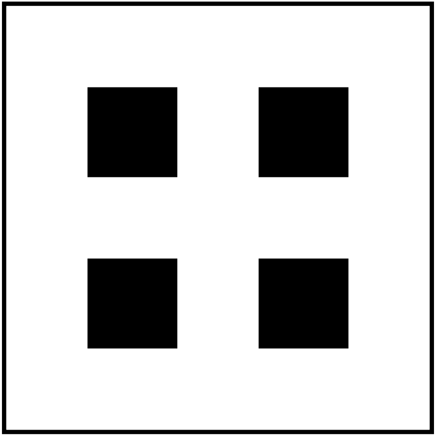
Sierpinski carpets

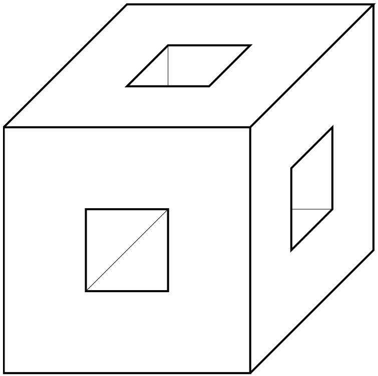
We consider bounded fractals such as the Sierpinski carpet.



Generalized Sierpinski carpets

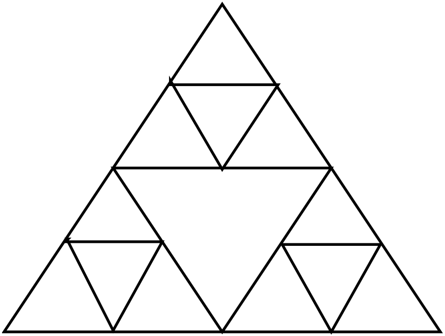


The Menger sponge



Sierpinski gaskets

Unlike the Sierpinski gasket, GSC's are not finitely ramified.



Existence

(a) Barlow-Bass (1989)

(b) Kusuoka-Zhou (1992)

(c) Barlow-Bass (1999)

See also Osada.

Also: Hambly, K. Hattori, T. Hattori, Hino, Hu, Pietruska-Pałuba, Stós, Watanabe, and others

The '89 paper had 4 questions, and one was added in the '99 paper:

(a) Higher dimensional SCs

(b) Local times (essentially, good estimates on heat kernels)

(c) Uniqueness

(d) Random state spaces

(e) (1999) Characterize the spectral dimension

On p. 256 of the '89 paper, last line, regarding uniqueness, "This problem seems quite hard."

Definitions

Let $\mathfrak{E}$ denotes the set of (i) non-zero; (ii) local; (iii) regular (iv) conservative Dirichet forms, which are (v) "invariant under all local symmetries."

Note elements of $\mathfrak{E}$ do not have to be scale invariant.

A Dirichlet form $\mathcal{E}$ with domain $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E})$ is

Symmetric: $\mathcal{E}(f, g) = \mathcal{E}(g, f)$.

Bilinear: $\mathcal{E}(af + g, h) = a\mathcal{E}(f, h) + \mathcal{E}(g, h)$.

Closed: The space $\mathcal{D}(\mathcal{E})$ with norm

$$\Big((f, f) + \mathcal{E}(f, f) \Big)^{1/2}$$

is complete.

Markovian: If

$$\varphi(t) = \begin{cases} t, & |t| \leq 1, \\ 1, & t > 1, \\ -1, & t < -1, \end{cases}$$

and $f \in \mathcal{D}(\mathcal{E})$, then $\varphi(f) \in \mathcal{D}(\mathcal{E})$ and

$$\mathcal{E}(\varphi(f), \varphi(f)) \leq \mathcal{E}(f, f).$$

Regular: If $\mathcal{C}$ is the set of continuous functions with compact support, then $\mathcal{D}(\mathcal{E}) \cap \mathcal{C}$ is dense in $\mathcal{C}$ with respect to the sup norm, and dense in $\mathcal{D}(\mathcal{E})$ with respect to the norm that we used to define what it means to be closed.

Conservative: If P_t is the associated semigroup, then $P_t 1 = 1$

Local: If f, g have disjoint supports and are in the domain of the Dirichlet form, then $\mathcal{E}(f, g) = 0$.

An example

Look at a domain in $\mathbb{R}^d$ and let

$$\mathcal{E}(f,g) = \frac{1}{2}\int_D \nabla f(x)\cdot\nabla g(x)\,dx,$$

where the domain is some suitable function space.

The domain makes a difference!

If one takes the infinitesimal generator of Brownian motion, namely, one half the Laplacian, then

$$\frac{1}{2}\int_D f(x)\Delta g(x)\,dx = -\frac{1}{2}\int_D \nabla f(x)\cdot\nabla g(x)\,dx = -\mathcal{E}(f,g).$$

I ignored the boundary term, a dangerous thing to do!

The main theorem

Theorem 1.

(a) $\mathfrak{E}$ is non-empty (in fact, the B-B and K-Z processes belong)

(b) Up to scalar multiples, $\mathfrak{E}$ consists of one element.

Corollaries

(a) $\mathcal{E}_{BB} = c\mathcal{E}_{KZ}$,

(b) The $\mathcal{E}_{BB}$ processes are scale invariant after all.

(c) There is a well-defined Laplacian on F.

A process formulation

Theorem 2. *Let X_t be (i) strong Markov; (ii) non-degenerate; (iii) continuous paths; (iv) state space F; (v) "invariant under all local symmetries." Then the law of X under $\mathbb{P}^x$ is uniquely defined.*

Two caveats:

Unique up to deterministic time change

There is always a null set of x's to deal with, one way or another.

What does invariant mean?

Suppose when constructing F, we divide the unit cube into m^d subcubes and remove some of them.

If S_1 and S_2 are subcubes constructed at the n^{th} stage, so that S_1, S_2 have side lengths m^{-n}, and both S_1, S_2 contain points of F, then the processes reflected on the boundaries of S_i, $i = 1, 2$, have the same law.

To be more precise, let R_S be the restriction operator, and U_S the unfolding operator. Define

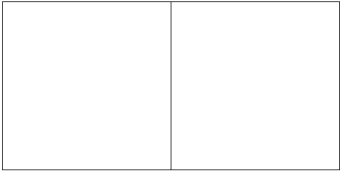
$$\mathcal{E}_S(g,g) = \frac{1}{m^n}\mathcal{E}(U_S g, U_S g).$$

Suppose $\Phi : S_1 \to S_2$ is an isometry. Then

$$\mathcal{E}_{S_1}(f \circ \Phi, f \circ \Phi) = \mathcal{E}_{S_2}(f,f)$$

and

$$\mathcal{E}(f,f) = \sum_S \mathcal{E}_S(R_S f, R_S f).$$



The example revisited

F is $[0,1]^2$ and we look at ordinary Brownian motion with reflection on the boundary of F. Then

$$\mathcal{E}(f,f) = \tfrac{1}{2} \int_F |\nabla f(x)|^2 \, dx,$$

and in this case

$$\mathcal{E}_S(f,f) = \tfrac{1}{2} \int_S |\nabla f(x)|^2 \, dx.$$

The condition $\mathcal{E}(f,f) = \sum_S \mathcal{E}_S(R_S f, R_S f)$ is there to guarantee that nothing surprising happens on the boundary.

Skeleton of the proof

If $\mathcal{A}, \mathcal{B} \in \mathfrak{E}$, let

$$\lambda = \sup\{r > 0 : \mathcal{A} \geq r\mathcal{B}\}.$$

Then

(a) $\mathcal{C} = \mathcal{A} - \lambda\mathcal{B} \in \mathfrak{E}$ (not quite true)

(b) All elements of $\mathfrak{E}$ have the same domain and are comparable.

(c) So $\mathcal{C} \geq \varepsilon\mathcal{B}$ for some ε. But then $\mathcal{A} \geq (\lambda + \varepsilon)\mathcal{B}$, a contradiction to the definition of λ.

First complication

$\mathcal{C}$ need not be closed. We look at

$$\mathcal{C}_\delta = (1+\delta)\mathcal{A} - \lambda\mathcal{B},$$

get estimates independent of δ, and that is good enough.

Where most of the work is

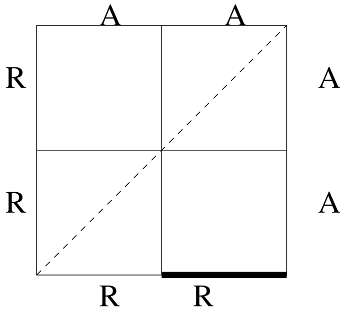
To show that all elements of $\mathfrak{E}$ have the same domain and are comparable, this comes down to heat kernel estimates for $p^{\mathcal{E}}(t,x,y)$ for arbitrary $\mathcal{E}$ in $\mathfrak{E}$. The ingredients necessary to getting heat kernel estimates are

(a) Corner moves and slides

(b) A coupling argument

(c) Elliptic Harnack inequality

(d) Resistance estimates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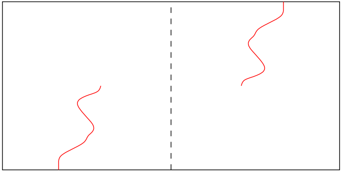

A major difficulty

What to do when starting in the middle of a L-shaped pattern. How do processes behave starting at such a point? The situation can be quite complicated in higher dimensions, but even in 2 dimensions it is a serious problem. This problem was avoided in Barlow-Bass because our approximations were reflecting Brownian motions, which do not hit points. For arbitrary $\mathcal{E}$ in $\mathfrak{E}$ we can't make any such approximation.